Bull. Cal.
Math. Soc

VOL. 75 198
NO. 1-6
G K. V

15

113619

[illegible] Binding House
Karachi [illegible] D.H.E.L. Road,
JAWALAPUR

Book Verification-20[illegible]

~~13~~

पुस्तकालय
गुरुकुल कांगड़ी विश्वविद्यालय
हरिद्वार

52

2/12/8

BULLETIN
OF THE
CALCUTTA MATHEMATICAL SOCIETY

COMPILED

VOLUME 75 **NUMBER 1**—4

No 1-6 95 c

FEBRUARY 1983

BOARD OF EDITORS

P. A. M. Dirac (U.K.)
A. N. Kolmogorov (U.S.S.R.)
I. M. Vinogradov (U.S.S.R.)
F. Harary (U.S.A.)
S. M. Shah (U.S.A.)
S. Dutta Majumdar (India)
N. D. Sengupta (India)
V. V. Narlikar (India)
A. K. Chaudhuri (India)

A. Jeffrey (U.K.)
B. D. Sharma (West Indies)
L. Debnath (U.S.A.)
B. R. Bhonsle (India)
C. K. Majumdar (India)
S. K. Chatterjea (India)
M. Dutta (India)
A. Ghosh (India)
S. P. Bandyopadhyay (India)

B. N. Mandal (Editorial Secretary)

Published by

CALCUTTA MATHEMATICAL SOCIETY
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA 9, INDIA.

THE COUNCIL OF THE CALCUTTA MATHEMATICAL SOCIETY

1983

President
B. R. Bhonsle

Vice-Presidents
F. Harary
L. Debnath
P. C. Vaidya
K. R. Unni
Shi I. Pai

Treasurer
S. K. Chatterjea

Secretary
M. Dutta

Editorial Secretary
B. N. Mandal

Assistant Secretary
U. Basu

Other Members of the Council
C. K. Majumdar
Arabinda Mukherjee
Netai De
B. D. Sharma
S. P. Chakraborty
Juthika Sengupta
N. D. Sengupta
S. P. Bandyopadhyay
B. K. Sengupta
J. Das
Radhikaranjan Pramanik
Salilesh Mukherjee

Benefactor : Calcutta University (India). East Carolina University (U.S.A.)

All correspondence with the society, subscriptions to the Bulletin, admission fees, and annual contributions of members are to be sent to the *Secretary* of the Society.

Paper intended for publication in the Bulletin of the Society and all Editorial correspondence should be addressed to the *Editorial Secretary* at the main office of the Society or to Professor L. Debnath at the branch office of the Society. Manuscripts offered for publication should be prepared in accordance with the instructions listed on the inside back cover of this issue.

Address of the Main Office : *92, Acharya Prafulla Chandra Road, Calcutta-9, INDIA.*
Address of the Branch Office : *Department of Mathematics, ECU, Greenville, N.C., 27834 U.S.A.*

The Bulletin is published Bi-monthly in February, April, June, August, October, December.
Subscription rate per volume for 1983 : Rs. 150/-(India) or $ 36.00 (Foreign).
Subscription rate per back volume : Rs. 420/- or $ 60.00

The publications of the Calcutta Mathematical Society may be purchased direct from the Society's office

113619

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 75 **Number 1** **February 1983**

COMPILED

113619

CONTENTS

PAGE

JAMES V. PETERS

The Müntz-szasz Theorem for Countable Sets ... 1

EVAGELOS PSOMOPOULOS

Some Commutativity Theorems for Rings and Groups 5

M. A. SOLIMAN AND N. H. ABDEL-ALL

Class of line complexes in the 2nd differential neighbourhood of the Ray in the flag space F_3 9

M. A. SOLIMAN AND N. H. ABDEL-ALL

Classification of line Complexes in the flag space F_3 17

ABOULHASSAN M. KHIDR

A Parallel to a Generalization of the Multinomial Formula 23

A. K. PRAMANIK

Generation of Internal Gravity Waves in a Stratified Ocean 27

M. A. SOLIMAN AND N. H. ABDEL-ALL

Classification of line complexes in the Ist differential neighbourhood of Ist ray in Isotropic space I_3 41

E. C. OBI

Some local properties of the Rayleigh Functions 49

JUTTA HAUSEN AND JOHNNY A JOHNSON

A new Characterization of perfect and semi-perfect Rings 57

GRIGORIOS TSAGAS

Relation between Topology and Properties of Curvature Operator of a Riemannian Manifold 59

JOHN GILL

Converging Factors for certain continued Fractions $K(a_n/1\}$, where $|a_n| < 1$. 63

M. A. ABDEL-GAID

Study of Turbulent Boundary layers on Porous Surface 69

M. A. KANDIL

Convergence of Fourier coefficients' series for vector valued Functions ... 77

Per volume: (For 1983) Rs. 150/- (India), or $ 36.00 (Foreign)
Subscription Rate per back volume : Rs. 420/- (India) or $ 60.00

PRINTED IN INDIA

PRINTED BY HARADHAN CHAKRABARTI, EKA PRESS, 204/1, B. T. ROAD, CALCUTTA-35
AND PUBLISHED BY M. DUTTA ON BEHALF OF CALCUTTA MATHEMATICAL SOCIETY,
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9.

CALCUTTA MATHEMATICAL SOCIETY

92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA.

NOTES FOR AUTHORS

On General Requirements

1. The Bulletin of the Calcutta Mathematical Society publishes papers presented to the society containing original works in different branches of mathematics and allied sciences,

2. Authors should send to the *Editorial Secretary* of the Society two copies of their communications, (one original copy) typed neatly on white paper, on one side only, with double spacing and with a margin of 4 c. m. together with all ancillary materials, while a further copy should be retained by them.

3, Authors will receive ordinarily one proof for corrections. Corrections to the proofs should be restricted to printer's errors only and made according to standard practice, Cost of any other corrections will be charged to the authors.

4. Authors are required to inform the Editorial Secretary of any change of address which occurs whilst their papers are in the process of publication.

5. The printing cost of the paper at the rate of Rs. 40/– or $ 10 per page should be paid by the Authors or their Institutions.

On Script Requirements

1. The first page should bear in order :—

The title, brief but informative ; the name(s) of the author(s) ; the address to which the proofs should be sent ; a running title of not more than 55 letters or signs ; a brief abstract followed by the introduction.

2. Equations and symbols may be handwritten or typewritten ; clarity for the printer is essential. Unusual symbols should be identified by name in the margin (in pencil) the first time they are used. If handwritten, care should be taken to distinguish between capital and lower case letters *e.g.* c, k, o, p, s, w, phi, psi ; between alpha (α) and a ; kappa (κ) and k ; mu (μ) and u ; nu (ν) and v ; eta (η) and n ; omega (ω) and w. When used as order symbol o and O should be shown as $\underset{-}{o}$ and $\underset{\equiv}{O}$.

Fractional exponents are to be used instead of root sign, $n!$ for $\underline{|n}$ exp ( ) notation when the exponent is complicated. Repetition of complicated expression is to be avoided by representing it with a symbol.

3. Reference to the literature in the text should be by author's surname and year in parentheses. References cited in the text should be listed at the end of the paper in the alphabatical order of the surname of the author in the following way (a recent number of the Bulletin may be a proper guide) :

Chandrasekhar, S. (1961), Hydrodynamic and Hydromagnetic Stability,
Oxford University press, London, p. 10.

Wilson, B.M. (1922), Proc. Lond. Math, Soc. (2), 21, 235.

4. Tables should be typed on pages separate from the text, numbered sequentially and planned to fit the page width of the journal (13 c.m.). Table numbers should be typed separately in sequence on the manuscript paper.

5. Line drawing must be clearly and sharply drawn with black India ink on white tracing paper. Figures should be drawn about 4 times the size which would appear in print. Standard size after reduction is normally 7 c.m, by 8 c.m. Each figure must have a number. Figure numbers should be typed in sequence separately on manuscript paper. The back of each figure should bear the name(s) of author(s) and the figure number.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 75 Number 2 April 1983

COMPILED

CONTENTS

PAGE

Tarit Kumar Mazumder
Graph Topologies on a Set 83

Anadi Kumar Chatterjee
Non-Linear Computational Model of a Tidal River Under the Action of Wind-Stress and Upland Flow 95

N. C. Das and S. K. Bhattacharya
Attenuation of Acoustic Plane Wave in a Conducting Cubic Crystal in a Magnetic Field 101

S ipati Ranjan Chaudhuri
Small Radial Vibration of Finitely Deformed Long Elastic Cylinder ... 109

B. M. Rastogi and S. C. Agrawal
The Stability of Vertically Stratified Non-Conducting Isentropic Shear Flow Between two Parallel Plates 117

Anadi Kumar Chatterjee and Sukomal Chandra Roy
Oscillation of a Body of Revolution in a Viscous Fluid 129

S. C. Gupta
Unsteady Channel Flow of a Conducting Fluid with Suspended Particles 137

L. Chatterjee and T. Roy
Scattering of Muons by a Screened Coulomb Potential 145

Subscription Rate per volume (For 1983) Rs. 150/- (India), or $ 36.00 (Foreign)
Subscription Rate per back volume : Rs. 420/- (India) or $ 60-00 (Foreign)

PRINTED IN INDIA

PRINTED BY DR. M. R. GUPTA AT ADBAR IMPRESSIONISTS, 34C, RAJA NABA KISSEN STREET, CALCUTTA-5
AND PUBLISHED BY PROF. MAHADEV DUTTA ON BEHALF OF CALCUTTA MATHEMATICAL SOCIETY.
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA

CALCUTTA MATHEMATICAL SOCIETY
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA

NOTES FOR AUTHORS

On General Requirements

1. The Bulletin of the Calcutta Mathematical Society publishes papers presented to the Society containing original works in different branches of mathematics and allied sciences.

2. Authors should send to the *Editorial Secretary* of the Society two copies of their communications, typed triplicate neatly on white paper, on one side only, with double spacing and with a margin of 4 cm. together with all ancillary materials, while a further copy should be retained by them.

3. Authors will receive ordinarily one proof for corrections. Corrections to the proof should be restricted to the printer's errors only and made according to standard practice. Cost of any other corrections will be charged to the authors.

4. Authors are required to inform the Editorial Secretary of any change of address which occurs whilst their papers are in the process of publication.

5. The printing cost of the paper at the rate of Rs. 40/- or $ 10 per page should be paid by the Authors or their Institution.

On Script Requirements

1. The first page should bear in order :—

The title, brief but imformative ; the name(s) of the author(s) ; the address to which the proofs should be sent ; a running title of not more than 55 letters or signs ; a brief abstract followed by the introduction.

2. Equations and symbols may be handwritten or typewritten ; clarity for the printer is essential. Unusual symbols should be identified by name in the margin (in pencil) the first time they are used. If handwritten, care should be taken to distinguish between capital and lower case letters *e. g.* c, k, o, p, s, w, phi, psi ; between alpha (α) and a ; kappa (k) and k ; mu (μ) and u ; nu (ν) and v ; eta (η) and v ; omega (ω) and w. When used as order symbol o and O should be shown as $\underline{\underline{o}}$ and $\underline{\underline{\underline{\underline{O}}}}$.

Fractional exponents are to be used instead of root sign, $n\,!$ for $\underline{|n}$, exp ( ) notation when the exponent is complicated. Repetition of complicated expression is to be avoided by representing it with a symbol.

3. References to the literature in the text should be by author's surname and year in parentheses. References cited in the text should be listed at the end of the paper in the alphabetical order of the surname of the author in the following way (a recent number of the Bulletin may be a proper guide) :

Wilson, B. M. (1922), *Proc. Lond. Math. Soc.* (2), 21, 235.

Chandrasekhar, S. (1961), Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London, p. 10.

4. Table should be typed on pages separate from the text, numbered sequentially and planned to fit the page width of the journal (13 cm.). Table numbers should be typed separately in sequence on the manuscript paper.

5. Line drawing must be clearly and sharply drawn with black India ink on white tracing paper. Figures should be drawn about 4 times the size which would appear in print. Standard size after reduction is normally 7 cm. by 8 cm. Each figure must have a number. Figure numbers should be typed in sequence separately on manuscript paper. The back of each figure should bear the name(s) of author(s) and the figure number.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 75 Number 3 June 1983

CONTENTS

COMPILED

PAGES

RANJIT KUMAR CHAKRABORTY

Steady State Solution of Some Heat Conduction Problems Under Mixed Boundary Conditions 153

DIPA ROY (Nèè Bose)

On Axiomatic Characterisation of n-th Order Information Improvement 161

ARATI BISWAS AND K. C. RAY

On Steinhaus Property for Distance Sets in a Metric Space 169

S. GHOSH

On a Theorem Concerning Lebesgue-Integration ... 175

S. K. SARASWAT AND S. K. GUPTA

Spaces of Strongly σ-Summable Sequences ... 179

SUJATA SINHA

An Algorithm for Network Decompositions ... 185

T. K. DUTTA

On Pseudo Ideal Series of Groups 191

P. MITRA

Unsteady Flow of a Dusty Viscous Incompressible Gas Through a Circular Tube Under Time Dependent Pressure Gradient 196

BHULU RANJAN SAHA

Decay of Eddy Turbulence Obtained Through Linearization of Dynamical Equations 205

Subscription Rate per volume (For 1983) Rs. 150. (India), or $ 36.00 (Foreign)
Subscription Rate per back volume ; Rs. 420- (India) or $ 60.00 (Foreign)

PRINTED IN INDIA
PRINTED BY M. R. GUPTA AT MUDRANIKA, 13A, BEPIN PAL ROAD, CALCUTTA-26
AND PUBLISHED BY MAHADEV DUTTA ON BEHALF OF THE CALCUTTA MATHEMATICAL SOCIETY.,
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA

CALCUTTA MATHEMATICAL SOCIETY
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA.

NOTES FOR AUTHORS

On General Requirements

1. The Bulletin of the Calcutta Mathematical Society publishes papers presented to the Society and containing original works in different branches of mathematics and allied sciences.

2. Authors should send to the *Editorial Secretary* of the Society two copies of their communications, (one original copy) typed neatly on white paper, on one side only, with double spacing and with a margin of 4 c.m. together with all ancillary materials, while a further copy should be retained by them.

3. Authors will receive ordinarily one proof for corrections. Corrections to the proofs should be restricted to printer's errors only and made according to standard practice. Cost of any other corrections will be charged to the authors.

4. Authors are required to inform the Editorial Secretary of any change of address which occurs whilst their papers are in the process of publication.

5. The printing cost of the paper at the rate of Rs. 40/- or $ 10 per page should be paid by the Authors or their Institution.

On Script Requirements

1. The first page should bear in order:—

The title, brief but informative; the name(s) of the author(s); the address to which the proofs should be sent; a running title of not more than 55 letters or signs; a brief abstract followed by the introduction.

2. Equations and symbols may be handwritten or typewritten; clarity for the printer is essential. Unusual symbols should be identified by name in the margin (in pencil) the first time they are used If handwritten, care should be taken to distinguish between capital and lower case letter *e. g.* c, k, o, p, s, w, phi, psi; between alpha (α) and a; kappa (κ) and k; mu (μ) and u: nu (ν) and v; eta (η) and n; omega (ω) and w. When used as order symbol o and O should be shown as $\underline{o}$ and $\underline{\underline{O}}$.

Fractional exponents are to be used instead of root sign. n! for $\lfloor\underline{n}$, exp ( ) notation when the exponent is complicated. Repetition of complicated expression is to be avoided by representing it with a symbol.

3. References to the literature in the text should be by author's surname and year in parentheses. Reference cited in the text should be listed at the end of the paper in the alphabetical order of the surname of the author in the following way (a recent number of the Bulletin may be a proper guide)

Wilson. B. M. (1922): *Proc. Lond. Math. Soc.* (2), 21, 235.

Chandrasekhar, S. (1961): Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London, p. 10.

4. Tables should be typed on pages separate from the text, numbered sequentially and planned to fit the page width of the journal (13 c.m.). Table numbers should be typed separately in sequence on the manuscript paper.

5. Line drawings must be clearly and sharply drawn with black India ink on white tracing paper. Figures should be drawn about 4 times the size which would appear in print. Standard size after reduction is normally 7 c.m. by 8 c.m. Each figure must have a number. Figure numbers should be typed in sequence separately on manuscript paper. The back of each figure should bear the name(s) of author(s) and the figure number.

—×—

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 75 Number 4 August 1983

CONTENTS

PAGES

TARIT KUMAR MAJUMDAR
Continuity and Fixed Image Characterisation in Graph Topological Spaces 218

M. D. CHATURVEDI
On The Absolute Cesàro Summability of a Series Associated with the Derived Fourier Series 223

BANDANA GHOSH
Group-Theoretic Study of Certain Generating Functions of Jacobi Polynomials 227

B. B. SHARMA
The Stability of the Disturbed Motion of a Twisted Strip Under Centrifugal and Aerodynamic Force Field 232

S. K. CHATTRRJEA AND T. D. BANERJEE
Two Parameter Lie Groups and Special Functions 239

N, P, S. BAWA AND R. C. VARMA
Fixed Point Theorems in Locally Uniform Convex Banach Space 244

RABINDRA KUMAR BHATTACHARYYA
On Wave Propagation in Random Magneto-Visco-Elastic Medium 249

TULSHIDAS BANERJEE
Extension of an Identity of Redei for Laguerre Polynomial 257

A. MOKID
A Few Remarks on the Continuity Condition for a Compressible Fluid in a State of Turbulence 259

PREM CHAND GUPTA
Unsteady Magneto-Hydrodynamic Flow Through Porous Media in a Channel Whose Cross-Section is a Circular Sector 264

RATHINDRANATH CHAKRABARTI
Singularities in a Two-Fluid Medium with Surface Tension at Their Surface of Seperation 271

Subscription Rate per volume (For 1983) Rs. 150. (India), or $ 36.00 (Foreign)
Subscription Rate per back volume ; Rs. 420. (India) or $ 60.00 (Foreign)

PRINTED IN INDIA
PRINTED BY M. R. GUPTA AT MUDRANIKA, 13A, BEPIN PAL ROAD, CALCUTTA-26
AND PUBLISHED BY MAHADEV DUTTA ON BEHALF OF THE CALCUTTA MATHEMATICAL SOCIETY.,
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA

CALCUTTA MATHEMATICAL SOCIETY
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA.

NOTES FOR AUTHORS

On General Requirements

1. The Bulletin of the Calcutta Mathematical Society publishes papers presented to the Society and containing original works in different branches of mathematics and allied sciences.

2. Authors should send to the *Editorial Secretary* of the Society two copies of their communications, (one original copy) typed neatly on white paper, on one side only, with double spacing and with a margin of 4 c.m. together with all ancillary materials, while a further copy should be retained by them.

3. Authors will receive ordinarily one proof for corrections. Corrections to the proofs should be restricted to printer's errors only and made according to standard practice. Cost of any other corrections will be charged to the authors.

4. Authors are required to inform the Editorial Secretary of any change of address which occurs whilst their papers are in the process of publication.

5. The printing-cost of the paper at the rate of Rs. 40/- or $ 10 per page should be paid by the Authors or their Institution.

On Script Requirements

1. The first page should bear in order:—

The title, brief but informative; the name(s) of the author(s); the address to which the proofs should be sent; a running title of not more than 55 letters or signs; a brief abstract followed by the introduction.

2. Equations and symbols may be handwritten or typewritten; clarity for the printer is essential. Unusual symbols should be identified by name in the margin (in pencil) the first time they are used. If handwritten, care should be taken to distinguish between capital and lower case letter *e. g.* c, k, o, p, s, w, phi, psi; between alpha (α) and a; kappa (κ) and k; mu (μ) and u: nu (ν) and v; eta (η) and n; omega (ω) and w. When used as order symbol o and O should be shown as $\underline{o}$ and $\underline{\underline{\underline{O}}}$.

Fractional exponents are to be used instead of root sign. $n!$ for $\lfloor\underline{n}$, exp ( ) notation when the exponent is complicated. Repetition of complicated expression is to be avoided by representing it with a symbol.

3. References to the literature in the text should be by author's surname and year in parentheses. Reference cited in the text should be listed at the end of the paper in the alphabetical order of the surname of the author in the following way (a recent number of the Bulletin may be a proper guide)

Wilson. B. M. (1922): *Proc. Lond. Math. Soc.* (2), 21, 235.

Chandrasekhar, S. (1961): Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London, p. 10.

4. Tables should be typed on pages separate from the text, numbered sequentially and planned to fit the page width of the journal (13 c.m.). Table numbers should be typed separately in sequence on the manuscript paper.

5. Line drawings must be clearly and sharply drawn with black India ink on white tracing paper. Figures should be drawn about 4 times the size which would appear in print. Standard size after reduction is normally 7 c.m. by 8 c.m. Each figure must have a number. Figure numbers should be typed in sequence separately on manuscript paper. The back of each figure should bear the name(s) of author(s) and the figure number.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 75 Number 5 October 1983

CONTENTS

PAGE

T. N. VERMA
Real Inversion Formula for the Distributional Generalized Laguerre Transform 283

R. S. SINHA and V. S. P. CHOWDHURY
Projective Motion in Recurrent Finsler Spaces 289

R. S. SINHA and S. A. FARUQI
Commutation Formulae for Non-Linear Connections in a Finsler Space ... 295

K. SANKARA RAO and J. PANDURANGAN
Flow Field Analysis on the Booster During Tandem Stage Separation ... 303

V. KARUNAKARAN
Curvature and Covering Theorems for Univalent Functions 317

SUTAPA MUKHERJEE and R. C. CHOUDHARY
Magnetohydrodynamic Boundary Layer with Suction Along a Porous Wall in Howarth's Flow — 321

T. K. DUTTA
Pseudo-Ideals in Some Finite Groups and in Finitely Generated Abelian Groups 331

PRAMOD KUMAR MITTAL
On Some Magnetohydrostatic Configurations in Parabolic Coordinates ... 339

G. N. CHANDWANI
Generating Functions for Jacobi Gegenbauer and Bedients Polynomials ... 353

Subscription Rate per volume (For 1983) Rs. 150/- (India), or $ 36.00 (Foreign)
Subscription Rate per back volume : Rs. 420/- (India) or $ 60-00 (Foreign)

PRINTED IN INDIA

PRINTED BY DR. M. R. GUPTA AT ADBAR IMPRESSIONISTS, 34C, RAJA NABA KISSEN STREET, CALCUTTA-5
AND PUBLISHED BY DR. U. BASU ON BEHALF OF CALCUTTA MATHEMATICAL SOCIETY,
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA

CALCUTTA MATHEMATICAL SOCIETY
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA

NOTES FOR AUTHORS

On General Requirements

1. The Bulletin of the Calcutta Mathematical Society publishes papers presented to the Society and containing original works in different branches of mathematics and allied sciences.

2. Authors should send to the *Editorial Secretary* of the Society two copies of their communications, one first copy typed triplicate neatly on white paper, on one side only, with double spacing and with a margin of 4 cm. together with all ancillary materials, while a further copy should be retained by them.

3. Authors will receive ordinarily one proof for corrections. Corrections to the proof should be restricted to the printer's errors only and made according to standard practice. Cost of any other corretions will be charged to the authors.

4. Authors are required to inform the Editorial Secretary of any change of address which occurs whilst their papers are in the process of publication.

5. The printing cost of the paper at the rate of Rs. 40/- or $ 10 per page should be paid by the Authors or their Institution.

On Script Requirements

1. The first page should bear in order :—

The title, brief but informative ; the name(s) of the author(s) ; the address to which the proofs should be sent ; a running title of not more than 55 letters or signs ; a brief abstract followed by the introduction.

2. Equations and symbols may be handwritten or typewritten ; clarity for the printer is essential. Unusual symbols should be identified by name in the margin (in pencil) the first time they are used. If handwritten, care should be taken to distinguish between capital and lower case letters *e. g.* c, k, o, p, s, w, phi, psi ; between alpha (α) and a ; kappa (k) and k ; mu (μ) and u ; nu (ν) and v ; eta (η) and n ; omega (ω) and w. When used as order symbol o and O should be shown as $\underset{-}{o}$ and $\underset{\equiv}{O}$.

Fractional exponents are to be used instead of root sign, n ! $\underline{|n}$, exp ( ) notation when the exponent is complicated. Repetition of complicated expression is to be avoided by representing it with a symbol.

3. References to the literature in the text should be by author's surname and year in parentheses. References cited in the text should be listed at the end of the paper in the alphabetical order of the surname of the authors in the following way (a recent number of the Bulletin may be a proper guide).

Wilson, B. M. (1922), *Proc. Lond. Math. Soc.* (2), **21**, 235.

Chandrasekhar, S. (1961), Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London, p. 10.

4. Table should be typed on pages separate from the text, numbered sequentially and planned to fit the page width of the journal (13 cm.). Table numbers should be typed separately in sequence on the manuscript paper.

5. Line drawing must be clearly and sharply drawn with black India ink on white tracing paper. Figures should be drawn about 4 times the size which would appear in print. Standard size after reduction is normally 7 cm. by 8 cm. Each figure must have a number. Figure numbers should be typed in sequence separately on manuscript paper. The back of each figure should bear the name(s) of author(s) and the figure number.

BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY

Volume 75 Number 6 December 1983

CONTENTS

PAGES

U. V. FATTEH and DARSHAN SINGH

A Note on D-Spaces 363

S. GHOSH

On a Theorem in Real Analysis 369

L. M. TRIPATHI and V. N. TRIPATHI

On Strong Perron Summability of the (c, 1) Mean of a Fourier Series 373

L. M. TRIPATHI and V. N. TRIPATHI

On Uniform Triangular Matrix Summability of Legendre Series 377

BHULU RANJAN SAHA and KSHETRA MOHAN GHOSH

Criteria Determining the Stability of the Quasi-Equilibrium Spectrum Envisazed Through Linearization of the Hydrodynamical Equation of Motion in Absence of Pressure-Gradient Term 381

BAL KISHAN DASS and VINOD TYAGI

Error Locating Codes with Sub-Blocks of Unequal Lengths 389

BHULU RANJAN SAHA and KSHETRA MOHAN GHOSH

"Diffusion Processes in a Homogeneous Shear-Free Turbulence Under the Assumption of Quasi-Normal Behaviour of Coupled Velocity—Pressure Field" 393

Subscription Rate per volume (For 1983) Rs. 150/- (India), or $ 36.00 (Foreign)
Subscription Rate per back volume : Rs. 420/- (India) or $ 60-00 (Foreign)

PRINTED IN INDIA

PRINTED BY DR. M. R. GUPTA AT ADBAR IMPRESSIONISTS, 34C, RAJA NABA KISSEN STREET, CALCUTTA-5
AND PUBLISHED BY DR. U. BASU ON BEHALF OF CALCUTTA MATHEMATICAL SOCIETY,
92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA

CALCUTTA MATHEMATICAL SOCIETY

92, ACHARYA PRAFULLA CHANDRA ROAD, CALCUTTA-9, INDIA

NOTES FOR AUTHORS

On General Requirements

1. The Bulletin of the Calcutta Mathematical Society publishes papers presented to the Society and containing original works in different branches of mathematics and allied sciences.

2. Authors should send to the *Editorial Secretary* of the Society two copies of their communications, one first copy typed triplicate neatly on white paper, on one side only, with double spacing and with a margin of 4 cm. together with all ancillary materials, while a further copy should be retained by them.

3. Authors will receive ordinarily one proof for corrections. Corrections to the proof should be restricted to the printer's errors only and made according to standard practice. Cost of any other corretions will be charged to the authors.

4. Authors are required to inform the Editorial Secretary of any change of address which occurs whilst their papers are in the process of publication.

5. The printing cost of the paper at the rate of Rs. 40/- or $ 10 per page should be paid by the Authors or their Institution.

On Script Requirements

1. The first page should bear in order :—

The title, brief but informative ; the name(s) of the author(s) ; the address to which the proofs should be sent ; a running title of not more than 55 letters or signs ; a brief abstract followed by the introduction.

2. Equations and symbols may be handwritten or typewritten ; clarity for the printer is essential. Unusual symbols should be identified by name in the margin (in pencil) the first time they are used. If handwritten, care should be taken to distinguish between capital and lower case letters *e. g.* c, k, o, p, s, w, phi, psi ; between alpha (α) and a ; kappa (κ) and k ; mu (μ) and u ; nu (ν) and v ; eta (η) and n ; omega (ω) and w. When used as order symbol o and O should be shown as o and $\underline{\mathrm{O}}$.

≡

Fractional exponents are to be used instead of root sign, n ! $\underline{|n}$, exp ( ) notation when the exponent is complicated. Repetition of complicated expression is to be avoided by representing it with a symbol.

3. References to the literature in the text should be by author's surname and year in parentheses. References cited in the text should be listed at the end of the paper in the alphabetical order of the surname of the authors in the following way (a recent number of the Bulletin may be a proper guide).

Wilson, B. M. (1922), *Proc. Lond. Math. Soc.* (2), 21, 235.

Chandrasekhar, S. (1961), Hydrodynamic and Hydromagnetic Stability, Oxford University Press, London, p. 10.

4. Table should be typed on pages separate from the text, numbered sequentially and planned to fit the page width of the journal (13 cm.). Table numbers should be typed separately in sequence on the manuscript paper.

5. Line drawing must be clearly and sharply drawn with black India ink on white tracing paper. Figures should be drawn about 4 times the size which would appear in print. Standard size after reduction is normally 7 cm. by 8 cm. Each figure must have a number. Figure numbers should be typed in sequence separately on manuscript paper. The back of each figure should bear the name(s) of author(s) and the figure number.

Bull. Cal. Math. Soc., **75**, 1–4 (1983)

THE MÜNTZ-SZASZ THEOREM FOR COUNTABLE SETS

JAMES V. PETERS

(*Received* 3 *November* 1981)

Abstract : We establish an analog of the Müntz-Szasz theorem for functions defined on closed countable subsets of the unit interval. As our theorems show, the hypotheses of the classical theorem may be weakened for these domains. Partial results for uncountable closed sets are also obtained.

According to the Weierstrass approximation theorem every continuous function on the unit interval $[0, 1]$ may be uniformly approximated by polynomials. Equivalently, the closed space of the set of functions $\{t^{\lambda}\}$ where λ ranges over the non-negative integers is $[0, 1]$. This suggests the following more general question. Given a sequence of non-negative real numbers $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ tending toward infinity, when does the closed space of $\{t^{\lambda}\}$ equal $C[0, 1]$? One necessary condition is that $\lambda_0 = 0$; this means that all the constant functions are in the closed span of $\{t^{\lambda}\}$. Beyond this condition, which will be assumed throughout the rest of the paper, the Müntz-Szasz theorem states that the closed span of $\{t^{\lambda_k}\}$ is $C[0, 1]$ if and only if

$$\sum_{k=1}^{\infty} 1/\lambda_k = \infty. \tag{1}$$

(See Goffman and Pedrick (1965) and Rudin (1966) for two entirely different proofs of this theorem). Setting $\lambda_k = k$ yields the Weierstrass theorem. A more interesting example is obtained by setting λ_k equal to the k-th prime. Since

$$\sum_{k=1}^{\infty} 1/p_k = \infty$$

where p_k denotes the k-th prime, the functions $1, t^2, t^3, \dots$ span $C[0, 1]$. On the other hand, setting λ_k equal to the k-th Fibonacci number does not yield a spanning set. Indeed, by the ratio test

$$\lim_{k\to\infty} \frac{1/F_{k+1}}{1/F_k} = \frac{2}{1+\sqrt{5}} < 1$$

where F_k denotes the k-th Fibonacci number. The same is true for $\lambda_k = r^k$ for any $r > 1$.

The problem of interest to us here is to obtain an analog of the Müntz-Szasz theorem for closed countable subsets of $[0, 1]$. Just as the Weierstrass theorem provides a sufficient (but not necessary) condition for the spanning of $C[0, 1]$, the Müntz-Szasz theorem yields a

---

Affiliation : C. W. Post Center of Long Island University Greenvale, New York 11548.

sufficient spanning condition for $C(E)$ for every closed $E \subset [0, 1]$: the validity of equation (1). As will be shown, weaker conditions are sufficient when E is countable.

It will be useful to introduce the integral transform

$$\hat{\mu}(\lambda) = \int_E t^\lambda d\mu(t)$$

where μ is a Borel measure on E. If $\hat{\mu}(\lambda) = 0$ for all $\lambda \in S$ then μ is said to be an annihilating measure of the closed span of

$$\{t^\lambda \mid \lambda \in S\}. \tag{2}$$

If, for a given set S, the only annihilating measure is $\mu \equiv 0$ then the closed span of (2) is $C(E)$.

THEOREM 1. *Given $E = \{t_1 t_2, \ldots, t_n\}$ the closed span of $\{t^\lambda\}$ is $C(E)$ for any n distinct values of λ.*

Proof : The only measures concentrated on finite sets are linear combinations of Dirac masses at each t_j. Thus, it must be shown that if

$$\sum_{j=1}^{n} \alpha_j t_j^\lambda = 0 \ (\alpha_j \text{ real}) \tag{3}$$

for n distinct values of λ, then each $\alpha_j = 0$. If one of the t_j's equals 0, then (3) may be reduced to a system of n equations in $n-1$ unknown α_j's The first of these equations has $\lambda = 0$. If the remaining $n-1$ equations imply that these $n-1$ α_j's vanish, then the first equation yields $\alpha_j = 0$ for $1 \leqslant j \leqslant n$. Consequently, there is no loss in generality in assuming that $0 \notin E$.

If $1 \notin E$ we put $\tau_\lambda = t_1^\lambda$ and choose positive constants p_j so that

$$\tau_\lambda^{p_j} = t_j^\lambda .$$

Making this substitution in (3), it follows that the function

$$\phi(\tau) = \sum_{j=1}^{n} \alpha_j \tau^{p_j}$$

has zeroes at each of n distinct values τ_λ. Since $\tau = 0$ is also a root, ϕ vanishes at $n+1$ distinct points. While ϕ is not necessarily a polynomial, it is still true that such a function cannot have $n+1$ roots unless it vanishes identically. This may easily be shown by successively dividing by each τ^{p_j}, then differentiating and applying Rolle's theorem.

If $1 \epsilon E$, say $t_n = 1$, then the above argument is modified to yield

$$\phi(\tau) = \sum_{j=1}^{n-1} \alpha_j \tau^{p_j} + \alpha_n.$$

The function ϕ has n distinct roots. It is easily verified that this function exhibits behavior similar to an $n-1$ degree polynomial in that the existence of n distinct roots implies that it vanishes identically. This completes the proof of the theorem.

An analysis of the proof of theorem 1 shows that the condition of n distinct values of λ satisfying (3) is necessary as well as sufficient. Thus, when E is infinite, the set of λ's necessary to span $C(E)$ must also be infinite. Recalling that the λ's were assumed to tend toward infinity, this condition is also sufficient.

THEOREM 2. *Given a closed set* $E = \{t_1, t_2, t_3, \ldots\}$ *the closed span of* $\{t^\lambda\}$ *is* $C(E)$ *for any infinite set* $\{\lambda_k\}$ *provided that* $\lambda_k \to \infty$.

Proof: As in theorem 1, it may be assumed without loss of generality that $0 \notin E$. Also, by analogy to theorem 1, it suffices to show that if

$$\sum_{j=1}^{\infty} \alpha_j t_j{}^\lambda = 0 \tag{4}$$

for each $\lambda = \lambda_k$ then all the α_j's vanish. Assume to the contrary that some $\alpha_j \neq 0$ and let t_s denote the supremum of the t_j's for which this is the case. Now fix $t_a < t_s$, put $\tau_\lambda = (t_a/t_s)^\lambda$ and choose positive constants p_j so that

$$\tau_\lambda{}^{p_j} = (t_j/t_s)^\lambda$$

for all $t_j < t_s$. Rewriting (4) in terms of this substitution yields

$$t_s{}^\lambda[\Sigma\alpha_j\tau_\lambda{}^{p_j} + \alpha_s] = 0.$$

where the summation is taken over all j corresponding to $t_j < t_s$. Since $t_s{}^\lambda \neq 0$ for any value of λ, the function

$$\phi(\tau) = \Sigma\alpha_j\tau^{p_j} + \alpha_s$$

has infinitely many roots τ_λ. As the λ's tend toward ∞, the τ_λ's tend toward 0. Since

$$\sum_{j=1}^{\infty} |\alpha_j| < \infty$$

and $0 < \tau < 1$, it follows that ϕ is continuous. This implies that

$$\alpha_s = \phi(0) = \lim_{\lambda \to \infty} \phi(\tau_\lambda) = 0$$

which contradicts our choice of t_s. Thus, all of the α_j's equal 0 and the proof is completed.

It would be of interest to extend the above results to uncountable subsets of $[0, 1]$. In this regard, two facts are evident. First, since the interval $[0, 1]$ may be replaced by any bounded interval $[a, b]$ in the Müntz-Szasz theorem, the validity of (1) is necessary whenever E has a non-void interior. Second, regardless of the structure of E, it follows from theorem 2 that there can be no discrete annihilating measure $\mu \not\equiv 0$.

References

Goffman, C. & Pedrick, G. (1965) : A First Course in Functional Analysis, Prentice-Hall, Englewood Cliffs, N. J.

Katznelson, Y. (1968) : An Introduction to Harmonic Analysis, John Wiley & Sons, N. Y.

Nevanlinna, R. and Paatero, V. (1969) : Introduction to Complex Analysis, Addison-Wesley, Reading, Mass.

Rudin, W. (1966) : Real and Complex Analysis, McGraw-Hill, N. Y.

C. W. POST CENTRE
LONG ISLAND UNIVERSITY
GREENVALE
NEW YORK 11548

Bull. Cal. Math. Soc., **75**, 5-8 (1983)

SOME COMMUTATIVITY THEOREMS FOR RINGS AND GROUPS

EVAGELOS PSOMOPOULOS

(*Received 4 June 1981*)

Abstract : In this note we prove commutativity theorems for rings satisfying certain polynomial identities. Also, we prove that the same theorems hold for groups too.

The following theorems are proved :

THEOREM 1. *Let n and m be relatively prime positive integers. Let R be an associative ring with identity* 1, *such that for all x, y in R* $(xy)^n = (yx)^m$. *Then R is commutative.*

THEOREM 2. *Let n, m and R be as in Theorem* 1. *If for all x, y in R,*

$$\text{(i) } [x^n, y] = [x, y^m], \text{ and (ii) } [x^n, y^m] = 0,$$

then R is commutative.

THEOREM 3. *Let n, m and R be as in Theorem* 1. *If for all x, y in R,*

$$\text{(i) } [x^n, y] = [x, y^m], \text{ and (ii) } [x^n, y^n] = 0,$$

then R is commutative.

THEOREM 4. *Let n, m and R be as in Theorem* 1. *If for all x, y in R,*

$$\text{(i) } [x^n, y] = [x, y^m] \text{ and (ii) } (xy)^n = (yx)^n,$$

then R is commutative.

The same theorems also hold for groups.

Groups. Throughout this section G will be a multiplicative group, and we write $[x, y] = xyx^{-1}y^{-1}$. Also, Z will denote the center of G.

THEOREM 1G. *Let G be a group such that for all x, y in G*

$$(xy)^n = (yx)^m \tag{1}$$

where n, m are relatively prime integers. Then G is abelian.

Proof. Replacing in (1), y by x^{n-1} and by x^{m-1}, we get $x^{n^2} = x^{nm}$ and $x^{m^2} = x^{nm}$.

Thus

$$x^{n^2} = x^{m^2}. \tag{2}$$

Now, replace in (1) again, x by $y^{-1}x^{n}$ to get

$$(y^{-1}x^{n}y)^{n} = (yy^{-1}x^{n})^{m},$$

and hence

$$y^{-1}x^{n^2}y = x^{nm} = x^{n^2}.$$

So,

$$x^{n^2}y = yx^{n^2}, \tag{3}$$

and thus by (2)

$$x^{m^2}y = yx^{m^2}. \tag{4}$$

Since n^2 and m^2 are relatively prime, there exist positive integers t and s such that $tm^2 - sn^2 = 1$. Take $N = sn^2$; then $1+N = tm^2$. Thus, (3) and (4) imply

$$x^{N}y = yx^{N} \text{ and } x^{N+1}y = yx^{N+1},$$

which in turn, implies

$$x^{N+1}y = yx^{N+1} = x^{N}yx.$$

Hence, after cancellation,

$$xy = yx. \tag{5}$$

This proves Theorem 1G.

THEOREM 2G. *Let n, m and G be as in Theorem 1G. If for all x, y in G,*

$$\text{(i) } [x^{n}, y] = [x, y^{m}], \textit{ and } \text{(ii) } [x^{n}, y^{m}] = 1,$$

then G is abelian.

Proof. In hypothesis (ii), replace first x by x^{m}, and then y by y^{n}, to get

$$[x^{mn}, y^{m}] = 1 \quad \text{and} \quad [x^{n}, y^{nm}] = 1$$

i.e.,

$$[x^{nm}, y^{m}] = 1 \quad \text{and} \quad [x^{nm}, y^{n}] = 1. \tag{6}$$

Now, the same method that gave (5) from (3) and (4) gives

$$x^{nm} \in Z \text{ for all } x \in G. \tag{7}$$

In hypothesis (i) replace first x by x^{m} and then y by y^{n} and use (7), to get

$$1 = [x^{nm}, y] = [x^{m}, y^{m}]$$

and

$$[x^{n}, y^{n}] = [x, y^{nm}] = 1.$$

The last two identities, $[x^n, y^n] = 1$ and $[x^m, y^m] = 1$, give commutativity of G by (Nicholson and Yaqub 1979)

THEOREM 3G. *Let n, m and G be as in Theorem 1G. If for all x, y in G,*

$$\text{(i) } [x^n, y] = [x, y^m], \text{ and (ii) } [x^n, y^n] = 1.$$

then G is abelian.

Proof. Replacing y by y^n in (i), and using (ii) we get

$$1 = [x^n, y^n] = [x, y^{nm}],$$

and hence

$$x^{mn} \in Z \text{ for all } x \in G. \tag{8}$$

Now, replace x by x^m in (i) and use (8) to get

$$1 = [x^{nm}, y] = [x^m, y^m]. \tag{9}$$

Therefore, hypothesis (ii) and (9) imply commutatively of G by (Nicholson and Yaqub 1979)

THEOREM 4G. *Let n, m and G be as in Theorem 1G. If for all x, y in G,*

$$\text{(i) } [x^n, y] = [x, y^m], \text{ and (ii) } (xy)^n = (yx)^n,$$

then G is abelian.

Proof. Replace x by $y^{-1}x$ in (ii) to get

$$(y^{-1}xy)^n = (yy^{-1}x)^n = x^n$$

$$y^{-1}x^n y = x^n$$

$$x^n y = yx^n. \tag{10}$$

Then hypothesis (i) gives

$$x^m \in Z \quad \text{for all } x \in G. \tag{11}$$

Now, the same method that gave (5) from (3) and (4), gives $xy = yx$ from (10) and (11).

Rings. Throughout this section R will be an associative ring with identity 1 and center Z; we write $[x, y] = xy - yx$.

Proof of Theorem 1. An inspection of the proof of Theorem 1G (see (3) and (4)) shows that the units of R are in Z, and thus the nilpotents are central. Replace y by 1 in $(xy)^n = (yx)^m$, to get

$$x^n = x^m.$$

Hence, R is commutative by Herstein's theorem [Herstein 1954].

The proofs of Theorems 2 and 3 are the same as those of Theorems 2G and 3G, respectively, in view of the following lemma which was proved in [Nicholson and Yaqub 1979 Lemma 4] :

LEMMA. *Let R be a ring with identity and let k be a positive integer. If for all x in R, $x^k[x, y] = 0$, then $[x, y] = 0$ for all x.*

Proof of Theorem 4. Since $y(xy)^n = (yx)^n y = (xy)^n y$, we have $[(xy)^n, y] = 0$. Replacing x by xy in (i), we get

$$0 = [(xy)^n, y] = [xy, y^m] = [x, y^m]y.$$

Also, by (i),

$$[x^n, y]y = [x, y^m]y = 0. \tag{12}$$

Replacing y by $y+1$ in (12) we have

$$[x^n, y](y+1) = 0.$$

which together with (12) gives

$$[x^n, y] = 0.$$

In particular, $[x^n, y^m] = 0$ and the theorem follows from Theorem 2.

References

Herstein, I. N. (1954) : "A note on rings with central nilpotent elements", *Proc. Amer. Math. Soc.*, 5, 620.

Nicholson, W. K. and Yaqub, A. (1979) : "A commutativity theorem for rings and groups", *Canad. Math., Bull.*, 22, 419.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF THESSALONIKI
THESSALONIKI, GREECE.

Bull. Cal. Math. Soc. 75, 9-15 (1983)

CLASS OF LINE COMPLEXES IN THE 2nd DIFFERENTIAL NEIGHBOURHOOD OF THE RAY IN THE FLAG SPACE F_3

M. A. SOLIMAN and N. H. ABDEL-ALL

(*Received* 14 *November* 1980)

Abstract : The present article is devoted to the differential geometry of line complexes whose the centres of the ray are the centres of inflection in the flag space F_3. For this class of line complexes, existence theorem is proved and its integral free-representation is given. The method adopted here is based on Cartan's differential calculus (Finikov 1948).

1. Introduction : The flag space F_3 regarded as a 3-dimensional projective space P_3 contains degenerate absolutum consisting of a plane with an invariant line in it and an invariant point on this line. The geometry under consideration, which is the differential geometry of a sixfold group in 3-dimensional space, shows much resemblance with the Euclidean geometry. In P_3 we use x^1, x^2, x^3, x^4 as homogeneous point coordinates. We consider the geometry belonging to the following sixfold group of projective transformations :

$$\begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \\ \bar{x}^4 \end{bmatrix} = \begin{bmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

Our geometry has the properties of the affine and therefore also of the projective geometry. As usual in affine geometry, we call the invariant plane $x^4=0$ the ideal plane and every configuration in this plane will be called ideal. For the points which do not lie in the ideal plane, we take $x^4=1$.

The distance of the ordered pair of points $P(x^1, x^2, x^3)$ and $Q(y^1, y^2, y^3)$ is y^3-x^3. If this distance is zero for two different points, we define y^2-x^2 as distance of second order and if this distance is zero too, we define y^1-x^1 as distance of third order. The angle between the ordered pair of planes $x^1+\xi_2x^2+\xi_3x^3+\xi_4x^4=0$ and $x^1+\eta_2x^2+\eta_3x^3+\eta_4x^4=0$ is given by $\xi_2-\eta_2$. If this angle is zero, we may introduce an angle of higher order as in the distance (Rosenfeld 1969).

We choose a moving frame conjugate to any arbitrary manifold immersed in the flag space F_3 as a coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, where the vertices $\bar{A}_1, \bar{A}_2, \bar{A}_3$ are the points at infinity, $\bar{A}_4$ is a proper point such that $\bar{A}_4\bar{A}_1, \bar{A}_4\bar{A}_2, \bar{A}_4\bar{A}_3$ form orthogonal traid, the invariant point is $\bar{A}_1$ and the invariant line $\bar{A}_1\bar{A}_2$ lie on the invariant plane

$\bar{A}_1\bar{A}_2\bar{A}_3$. The absolutum in the mentioned moving frame T consists of the plane $x^4 = 0$, the line $x^3 = 0$, $x^4 = 0$ and the point $(1, 0, 0, 0)$. The geometry is completely self-dual (Redie 1968).

The fundamental equations of the moving frame T are

$$d\begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega^1{}_2 & 0 & 0 & 0 \\ \omega^1{}_3 & \omega^2{}_3 & 0 & 0 \\ \omega^1{}_4 & \omega^2{}_4 & \omega^3{}_4 & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix}, \tag{1}$$

and the structural equations (the integrability conditions) are

$$D\begin{bmatrix} \omega^1{}_2 \\ \omega^1{}_3 \\ \omega^2{}_3 \\ \omega^1{}_4 \\ \omega^2{}_4 \\ \omega^3{}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^2{}_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^2{}_4 & \omega^3{}_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^3{}_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Lambda \begin{bmatrix} \omega^1{}_2 \\ \omega^1{}_3 \\ \omega^2{}_3 \\ \omega^1{}_4 \\ \omega^2{}_4 \\ \omega^3{}_4 \end{bmatrix}, \tag{2}$$

where $\omega^j{}_i$ are Pffaf's differential form, D denotes the exterior differentiation operator and Λ the exterior product between the differential forms.

9. Arbitrary Line complex Related to a Canonical Frame in F_3 : Let, in F_3, u, v, w be a system of curve linear coordinates; the admissible variables (u, v, w) are taken from an open neighbourhood of C^3. Consider a line complex immersed in F_3, which generated by the ray $l = (\bar{A}_3\bar{A}_4)$, $l = l(u, v, w)$, we denote by $(\bar{A}_3\bar{A}_4)$ the Grassmann coordinates of the line l. A coordinate tetrahedron $T(\bar{A}_1\,\bar{A}_2, \bar{A}_3, \bar{A}_4)$ with the coordinate planes $(\bar{A}_3\bar{A}_4\bar{A}_2)$ and $(\bar{A}_3\bar{A}_4\bar{A}_1)$ concident with the planes corresponding to the vertices $\bar{A}_3$ and $\bar{A}_4$ respectively in the normal correlation of the ray l is called a canonical tetrahedron and the vertices $\bar{A}_3$ and $\bar{A}_4$ are called the centres of the ray l of line complex (Kovansov 1963). The differential equation of this complex related to its canonical frame in the first differential neighbourhood of the ray l has the form

$$\omega^1{}_4 = k\omega^2{}_3 \tag{3}$$

where k is an invariant of the first differential neighbourhood and is called the curvature of the line complex. Excludes from our consideration the holonomic complex $\omega^2{}_3=0$. Exterior differentiation of the equation (3) and using Cartan's lemma we have

$$\begin{bmatrix} \omega^1{}_2 \\ -\omega^3{}_4 \\ dk \end{bmatrix} = \begin{bmatrix} p & \alpha & \beta \\ \alpha & q & \gamma \\ \beta & \gamma & r \end{bmatrix} \begin{bmatrix} \omega^2{}_4 \\ \omega^1{}_3 \\ \omega^2{}_3 \end{bmatrix}, \tag{4}$$

where $p, \alpha, \beta, q, \gamma, r$, are the invariants of the 2nd differential neighbourhood of the ray l and the forms $\omega^2{}_4$, $\omega^1{}_3$ and $\omega^2{}_3$ are basis forms which is linear combination of the differentials du, dv and dw.

LEMMA 1. *Any arbitrary line complex immersed in the flag space F_3 can be stratified into one-parameter family of holonomic line congruences.*

Proof: For the line complex (4), the differential equation

$$\omega^2{}_3=0 \tag{5}$$

automatically satisfies $D\,\omega^2{}_3\equiv 0$; then the equation (5) determines a holonomic line congruence belonging to the arbitrary line complex (4). It follows that, this complex can be distributed into one-parameter family of line congruences (5).

To find the focal points of the line congruence (5), let $\overline{F}=\bar{A}_3+t\bar{A}_4$ be any analytic points on the ray $l=(\bar{A}_3\bar{A}_4)$. This point will be a focal point if $d\overline{F}\equiv 0$, (mod $\overline{F}$) which gives the equations

$$\omega^1{}_3+t\omega^1{}_4=0,\ \omega^2{}_3+t\omega^2{}_4=0,\ (\omega^1{}_4=0=\omega^2{}_3).$$

This means that, the focal points corresponding to $t=0$, $t=\infty$ are the centres $\bar{A}_3$, $\bar{A}_4$ of ray l respectively. From the differentials

$$d\bar{A}_3=\omega^1{}_3\bar{A}_1,\ d(\bar{A}_3\bar{A}_1)=0,\ d\bar{A}_4=\omega^2{}_4\ (\bar{A}_2-\alpha\bar{A}_3)-q\omega^1{}_3\bar{A}_3,$$

we see that, the focal surfaces described by the focal points $\bar{A}_3$, $\bar{A}_4$ are the fixed line $(\bar{A}_3\bar{A}_1)$, arbitrary surface σ with tangent plane $(\bar{A}_3\bar{A}_4\bar{A}_2)$ respectively. The tangent plane to the focal surface σ coincident with the plane corresponding to the centre $\bar{A}_3$ of the ray l in the normal correlation.

3. **Existence theorem :** The centres of inflection of the ray of a given complex are defined in (Kovansov 1963) as these points having the property that, their corresponding line cone possesses this ray as a singular generator. The point $\overline{M}=\bar{A}_3+t\bar{A}_4$ will be a centre of inflection of the ray l if its abscissa satisfies the equation

$$qk^2t^4-2k\gamma t^3+(2k\alpha+r)t^2-2\beta t+p=0. \tag{6}$$

This means that, there are four centres of inflection on the ray l of the line complex (4).

If the centres $\bar{A}_3$ $(t=0)$ and $\bar{A}_4$ $(t=\infty)$ of the ray l of the complex (4) are centres of inflection, then $t=0$ and $t=\infty$ must satisfy equation (6). Thus we have $p=q=0$, and the differential equations of a line complex for which the centres of the ray l are centres of inflection are

$$\begin{bmatrix} \omega^1{}_4 \\ \omega^1{}_2 \\ -\omega^3{}_4 \\ dk \end{bmatrix} = \begin{bmatrix} 0 & 0 & k & 0 \\ 0 & \alpha & \beta & 0 \\ \alpha & 0 & \gamma & 0 \\ \beta & \gamma & r & 0 \end{bmatrix} \begin{bmatrix} \omega^2{}_4 \\ \omega^1{}_3 \\ \omega^2{}_3 \\ 0 \end{bmatrix}. \tag{7}$$

Exterior differentiation of the last three equations of (7) and using Cartan's lemma, we get

$$d\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ r \end{bmatrix} = \begin{bmatrix} 0 & 0 & \lambda_1-\alpha^2 & 0 \\ 0 & \lambda_1 & \lambda_2 & 0 \\ \lambda_1 & 0 & \lambda_3 & 0 \\ \lambda_2+\alpha\beta & \lambda_3+\alpha\gamma & \lambda_4 & 0 \end{bmatrix} \begin{bmatrix} \omega^2{}_4 \\ \omega^1{}_3 \\ \omega^2{}_3 \\ 0 \end{bmatrix}. \tag{8}$$

From (7) and (8) it follows that, the number of independent parameter $N=4$ $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. Using Cartan's common method, we have the number of independent characteristic forms $\bar{q}=4$ $(d\alpha, d\beta, d\gamma, dr)$, the number of independent quadratic exterior forms $S_1=3$, and the equality $\bar{q}=S_1+S_2$, gives that $S_2=1$. The Cartan's number $Q=S_1+2S_2=5$ is greater than the number of independent parameters. This means that, the system of differential equations (7) is not in involution.

For this reason we consider the system of differential equations (8). Exterior differentiation of (8) and using Cartan's lemma, we obtain

$$d\begin{vmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{vmatrix} = \begin{vmatrix} 0 & 0 & \mu_1 & 0 \\ 0 & \mu_1+\alpha\lambda_1 & \mu_2 & 0 \\ \mu_1+\alpha\lambda_1 & 0 & \mu_3 & 0 \\ \mu_2+\beta\lambda_1+2\alpha\lambda_2 & \mu_3+2\alpha\lambda_3+\gamma\lambda_1 & \mu_4 & 0 \end{vmatrix} \begin{vmatrix} \omega^2{}_4 \\ \omega^1{}_3 \\ \omega^2{}_3 \\ 0 \end{vmatrix}. \tag{9}$$

From (9) it follows that the number of independent parameter $N=4$ $(\mu_1, \mu_2, \mu_3, \mu_4)$, $\bar{q}=4$ $(d\lambda_1, d\lambda_2, d\lambda_3, d\lambda_4)$, $S_1=4$ and the equality $\bar{q}=S_1+S_2$, gives $S_2=0$ (the oldest character). The Cartan's number $Q=S_1=4=N$ the number of independent parameters. This means that, the system of differential equations (7) is in involution and exists within four functions of one argument. Hence we have the following existence theorem.

THEOREM 1. *The line complex for which the centres of the ray are centres of inflection exists within four arbitrary functions of one argument.*

4. The geometrical construction : The following theorems give the integral free-representation of the line complex (7).

THEOREM 2. *If the ray* $l = (\bar{A}_3\bar{A}_4)$ *generates the line complex* (7), *then the centre of the ray,* $\bar{A}_4$, *describes a ruled surface whose generator is* $m = (\bar{A}_4, \bar{A}_2-\alpha\bar{A}_3)$.

Proof : From the differentials

$$d\bar{A}_4 = \omega^2{}_3\{k\bar{A}_1-\gamma\bar{A}_3\}+\omega^2{}_4\{\bar{A}_2-\alpha\bar{A}_3\},$$

$$dm \equiv \omega^2{}_3\{k(\bar{A}_1, \bar{A}_2-\alpha\bar{A}_3)+(\bar{A}_4, \beta\bar{A}_1-\alpha\bar{A}_2-(\lambda_1-\alpha^2)\bar{A}_3)\},$$

it follows that, $d\bar{A}_4$ depends on two principal forms $\omega^2{}_3$, $\omega^2{}_4$ and dm depends on one principal form $\omega^2{}_3$, then the centre of the ray $\bar{A}_4$ describes a ruled surface Σ with generator m. The ruled surface Σ has tangent plane Σ with tangential coordinates.

$$\sigma \equiv (\bar{A}_4\bar{A}_2\bar{A}_1)-\alpha(\bar{A}_4\bar{A}_3\bar{A}_1)+\gamma/k(\bar{A}_4\bar{A}_3\bar{A}_2).$$

From the above theorem, we have the geometrical meaning of the invariants α and γ in the 2nd differential neighbourhood, the invaiant $-\alpha$ equal to the angle between σ and the coordinate plane $x^3 = 0$ and the invariant γ/k equal to the angle of 2nd order between σ and the coordinate plane $x^3 = 0$.

LEMMA 2. *The partial differential equation* $\omega^1{}_4 = 0$ *determines a hyperbolic linear line congruence with two directrices.*

Proof : For the line complex (7), the equation $\omega^1{}_4 = 0$ satisfies $D\omega^1{}_4 \equiv 0 \pmod{\omega^1{}_4}$, i.e., $\omega^1{}_4 = 0$ determines a holonomic congruence belonging to the line complex (7). The focal points of this line congruence are the centres of the ray $l = (\bar{A}_3\bar{A}_4)$. From the differentials

$$d\bar{A}_3 = \omega^1{}_3\bar{A}_1,\ d\bar{A}_4 = \omega^2{}_4\{\bar{A}_2-\alpha\bar{A}_3\},\ (\omega^2{}_3 = 0 = \omega^1{}_4)$$

and

$$d(\bar{A}_3\bar{A}_1) = 0,\ dm = d(\bar{A}_4, \bar{A}_2-\alpha\bar{A}_3) = 0,$$

it follows that, the focal points $\bar{A}_3$ and $\bar{A}_4$ describe the fixed lines $(\bar{A}_3\bar{A}_1)$ and m respectively. Hence the line congruence $\omega^1{}_4 = 0$, is a hyperbolic linear congruence with m and $(\bar{A}_3\bar{A}_1)$ as the two directrices.

COROLLARY. *The line complex (7) admits a fibration into one-parameter family of a hyperbolic linear line congruence.*

From

$$d\{\bar{A}_3+t_1\bar{A}_1\} = \{\omega^1{}_3+dt_1\}\bar{A}_1,\ D\{\omega^1{}_3+dt_1\} \equiv 0 \pmod{\omega^1{}_4},$$

$$d\{\bar{A}_4+t_2(\bar{A}_2-\alpha\bar{A}_3)\} = \{\omega^3{}_4+dt_2\}\{\bar{A}_2-\alpha\bar{A}_3\},\ D\{\omega^2{}_4+dt_2\} \equiv 0 \pmod{\omega^1{}_4},$$

it follows that, $t_1 = f(\tau)$, $t_2 = g(\tau)$ and any point on the directrix $(\bar{A}_3\bar{A}_1)$ of the linear congruence $\omega^1{}_4 = 0$, determines some point on the second directrix $m = (\bar{A}_4, \bar{A}_2-\alpha\bar{A}_3)$. This means that any point on the line $(\bar{A}_3\bar{A}_1)$ depends on the parameter τ of the ruled surface generated by m (i.e., depends on one arbitrary function of one parameter),

The following theorem gives the integral free-representation of the line complex (7).

THEOREM 3. *We consider a ruled surface. Construct a hyperbolic linear congruence, for which the generator of the ruled surface and a line on the ideal plane are directrices. All these congruences construct the line complex under discussion.*

Proof: We choose a frame of reference in the form of a coordinate canonical tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, such that the vertices $\bar{A}_1$ and $\bar{A}_3$ are placed on a generator of the ruled surface, this generator intersects the ideal plane at a point $\bar{P}$. On the line $(\bar{A}_3\bar{P})$ we place the point $\bar{A}_2$, in this case $\bar{P} = \bar{A}_2-\alpha\bar{A}_3$. The line $(\bar{A}_3\bar{A}_1)$ and $(\bar{A}_4\bar{P})$ will be the directrices of a hyperbolic linear congruence.

The line $(\bar{A}_1\bar{A}_3)$ is a directrix of a linear congruence generated by $l = (\bar{A}_3\bar{A}_4)$ if and only if $\omega^2{}_3 = 0$ and $\bar{A}_4$ describes the generator $m = (\bar{A}_4, \bar{A}_2-\alpha\bar{A}_3)$ if and only if $\omega^1{}_4 = 0$. If $l = (\bar{A}_3\bar{A}_4)$ coincident with the ray of the line complex, the principal forms of the moving frame will be $\omega^1{}_4$, $\omega^2{}_3$, $\omega^1{}_3$ and $\omega^2{}_4$. Since $\omega^1{}_4 = 0$ if and only if $\omega^2{}_3 = 0$, then the forms $\omega^1{}_4$ and $\omega^2{}_3$ must be related by

$$\omega^1{}_4 = k\omega^2{}_3. \tag{10}$$

Exterior differentiation of (10) leads to (4). The line m is a generator of the ruled surface and a directrix of linear congruence $\omega^2{}_3 = 0$ if and only if

$$dm \equiv \omega^2{}_3(k\bar{A}_1-\gamma\bar{A}_3, \bar{A}_2-\alpha\bar{A}_3)+(\bar{A}_4, \{\omega^1{}_2-\alpha\omega^1{}_3\}\bar{A}_1-\alpha\omega^2{}_3\bar{A}_2-d\alpha\bar{A}_3)$$

depends on the principal form $\omega^2{}_3$. This leads to

$$d\alpha\Lambda\omega^2{}_3 = 0,\ \{\omega^1{}_2-\alpha\omega^1{}_3\}\ \Lambda\ \omega^2{}_3 = 0. \tag{11}$$

From (4) using equations (11), we have

$$q = P = 0. \tag{12}$$

The relation (12) characterizes the constructed line complex (7).

References

Finikov, S. P. (1948) : Cartan's methods of exterior forms and their applications in differential geometry, Moscow.

Kovansov, N. I. (1963) : Theory of Complexes. Kiev.

Redie, L. (1968) : Foundation of Euclidean and non-Euclidean geometries.

Rosenfeld, B. A. (1969) : Non Euclidean geometry. Moscow.

DEPT. OF MATHS,
FACULTY OF SCIENCE,
ASSIUT UNIVERSITY,
ASSIUT, (EGYPT).

Bull. Cal. Math. Soc., 75, 17–22 (1983)

CLASSIFICATION OF LINE COMPLEXES IN THE FLAG SPACE F_3

M. A. Soliman and N. H. Abdel-All

(*Received* 30 *January* 1981)

Abstract : In the present paper, a complete classification of line complexes in the 3rd differential neighbourhood of the ray in the flag space F_3, based on the symmetry of the matrices which arise from the integrability conditions of the system of differential equations of a line complex, is given. Mainly, two subclasses are considered for each one the existence theorem is proved and the integral-free representation is given. The methods adopted are based on Cartan's differential calculus (Finikov 1948).

Introduction : A homogeneous space $F_3 \equiv (P_3, G_6)$ is called flag space if P_3 is 3-dimensional projective space with metric and G_6 is a six-fold subgroup of the group of projective transformations of P_3 that has degenerate absolutum consisting of a plane at infinity with an invariant line and invariant point on it (Penner 1967).

We choose a moving frame conjugate to any arbitrary manifold immersed in the flag space F_3 as a coordinate tetrahedron T $(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, where the vertices $\bar{A}_1, \bar{A}_2, \bar{A}_3$ are the points at infinity and $\bar{A}_4$ is a proper point such that the edges $\bar{A}_4\bar{A}_1$, $\bar{A}_4\bar{A}_2$, $\bar{A}_4\bar{A}_3$ form orthogonal traid. The invariant point $\bar{A}_1$ and the invariant line $\bar{A}_1\bar{A}_2$ lie on the invariant plane $\bar{A}_1\bar{A}_2\bar{A}_3$ (the ideal plane) (Redei 1968).

The fundamental equations of the moving frame T are

$$d \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ \omega^1_2 & 0 & 0 & 0 \\ \omega^1_3 & \omega^2_3 & 0 & 0 \\ \omega^1_4 & \omega^2_4 & \omega^3_4 & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix} \tag{1}$$

and the structural equations (i.e., the integrability conditions) are

$$D \begin{bmatrix} \omega^1_2 \\ \omega^1_3 \\ \omega^2_3 \\ \omega^1_4 \\ \omega^2_4 \\ \omega^3_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^2_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \omega^2_4 & \omega^3_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^3_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Lambda \begin{bmatrix} \omega^1_2 \\ \omega^1_3 \\ \omega^2_3 \\ \omega^1_4 \\ \omega^2_4 \\ \omega^3_4 \end{bmatrix} \tag{2}$$

where ω_i^j are Pfaff's differential form, D denotes the exterior differentiation operator and Λ the exterior product between the differential forms.

The differential equations of a line complex immersed in F_3, which is generated by the ray $l \equiv (\bar{A}_3\bar{A}_4)$, related to its canonical frame in the 2*nd* differential neighbourhood of its ray (Soliman 1980) are

$$\begin{bmatrix} \omega^1{}_4 \\ \omega^1{}_2 \\ -\omega^3{}_4 \\ dk \end{bmatrix} = \begin{bmatrix} 0 & 0 & k & 0 \\ p & \alpha & \beta & 0 \\ \alpha & q & \gamma & 0 \\ \beta & \gamma & r & 0 \end{bmatrix} \begin{bmatrix} \omega^2{}_4 \\ \omega^1{}_3 \\ \omega^2{}_3 \\ 0 \end{bmatrix} \tag{3}$$

where k is an invariant of the 1*st* differential neighbourhood and is called the curvature of the line complex; $p, \alpha, \beta, q, \gamma, r$ are the invariants of the 2*nd* differential neighbourhood of the ray.

Exterior differentiations of the Pfaffian system (3) and using Cartan's lemma, we get the following systems of equations

$$d[p\alpha\beta]^t = M_1\Omega,\ d[\alpha q\gamma]^t = M_2\Omega,\ d[\beta\gamma r]^t = M_3\Omega \tag{4}$$

where t denotes the transpose of a row matrix,

$$\Omega = \begin{bmatrix} \omega^2{}_4 \\ \omega^1{}_3 \\ \omega^2{}_3 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3-\alpha p \\ \lambda_2 & \lambda_4 & \lambda_5-\alpha^2 \\ \lambda_3+\alpha p & \lambda_5+pq & \lambda_6 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \mu_1 & \mu_2 & \mu_3-pq \\ \mu_2 & \mu_4 & \mu_5-\alpha q \\ \mu_3+\alpha^2 & \mu_5+\alpha q & \mu_6 \end{bmatrix}, \quad M_3 = \begin{bmatrix} \nu_1 & \nu_2 & \nu_3-p\gamma \\ \nu_2 & \nu_4 & \nu_5-\alpha\gamma \\ \nu_3+\alpha\beta & \nu_5+\beta q & \nu_6 \end{bmatrix} \tag{5}$$

where $\lambda_1, \ldots, \lambda_6$; $\mu_1, \ldots, \mu_6$ and $\nu_1, \ldots, \nu_6$ are the invariants in the 3*rd* differential neighbourhood of the ray.

In 1970, Orehova has introduced the notation of line complex with skew symmetric matrix in Euclidean space (Orehova 1970). We give therein a complete classification of line complexes in the space F_3 for which both M_1 and M_2 are symmetrical matrices whereas M_3 is not. The matrix M_1 is symmetric if and only if the matrix M_2 is symmetric and this is possible only if one of the following conditions (*a*) $\alpha = q = 0$, (*b*) $\alpha = p = 0$, is satisfied. We denote by $(K)_1$, $(K)_2$ and $(K)_{12}$ classes of line complexes for which M_1; M_2 and M_1, M_2 are symmetrical matrices respectively.

In this classification, the classes $(K)_1$, $(K)_2$ and $(K)_{12}$ are the same, we denote them by $(K)_I$. The class $(K)_I$ contains two subclasses ${}^a(K)_I$, ${}^b(K)_I$ according to the conditions (a), (b) respectively.

1. The subclass of line complexes ${}^a(K)_I$

Any line complex $K \epsilon {}^a(K)_I$ is characterized by the system of diffefential equations (3) with the condition $\alpha = q = 0$ and the system of differential equations

$$d \begin{bmatrix} p \\ \beta \\ \gamma \\ r \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \lambda_3 & 0 \\ \lambda_3 & 0 & \lambda_6 & 0 \\ 0 & 0 & \mu_6 & 0 \\ \lambda_6 & \mu_6 & \nu_6 & 0 \end{bmatrix} \begin{bmatrix} \omega^2{}_4 \\ \omega^1{}_3 \\ \omega^2{}_3 \\ 0 \end{bmatrix} \tag{6}$$

in the 3*rd* differential neighbourhood of the ray. For the line complex $K \epsilon {}^a(K)_I$, $\bar{A}_4$, the centre of the ray, is a centre of inflection (Kovansov 1963).

From (3) and (6), it follows that, the number of independent parameters $N = 5\,(\lambda_1, \lambda_3, \lambda_6, \mu_6, \nu_6)$. Using Cartan's method, we have the number of characteristic forms $\tilde{q} = 4\,(dp, d\beta, d\gamma, dr)$, the number of independent exterior forms $S_1 = 3$, and the equality $\tilde{q} = S_1 + S_2$ gives that $S_2 = 1$. Hence the Cartan's number $Q = S_1 + 2S_2 = 5 = N$. This means that, the system of differential equations (6) is in involution and exists within one arbitrary function of two arguments. Then, we have the following existence theorem.

THEOREM (1.1) : *The range of the existence of every line complex belonging to the subclass ${}^a(K)_I$ comprises one arbitrary function of two arguments.*

Geometrical properties of the subclass ${}^a(K)_I$:

We investigate some geometrical properties of the line complexes $K \epsilon {}^a(K)_I$, which can help us to give their integral-free representation (i.e., geometrical construction) (Veselova 1975).

Property (1.1) : The line complex $K \epsilon {}^a(K)_I$ is stratified into one parameter family of hyperbolic holonomic line congruences ${}^aC_1$, $\omega^2{}_3 = 0$, with focal surfaces degenerate to the ideal line $\bar{A}_3\bar{A}_1$ and a plane curve which are described by the focal points $\bar{A}_3$ and $\bar{A}_4$ respectively.

Property (1.2) : The line complex $K \epsilon\, {}^a(K)_I$ is stratified into one parameter family of hyperbolic holonomic line congruences ${}^aC_2$, $p\omega^2{}_4+\beta\omega^2{}_3 = 0$, with a space curve and developable surface as the focal surfaces described by the focal points $\bar{A}_4$ and $\bar{A}_4+(\beta/p)\bar{A}_3$ respectively.

The integral-free representation of the subclass ${}^a(K)_1$:

The meaning of integral-free representation of a line complex is to obtain its system of differential equations from its geometrical construction (Kovansov 1975).

THEOREM (1,2) : *Considering an arbitrary surface Σ_a, the lines drawn from every point belonging to this surface will meet the ideal plane. The family of all these lines construct any line complex $K \epsilon\, {}^a(K)_I$.*

Proof : We place the proper vertex $\bar{A}_4$ of the canonical coordinate tetrahedron T $(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$ on the surface Σ_a. The vertices $\bar{A}_1, \bar{A}_2, \bar{A}_3$ form the ideal plane. The tangent plane of the surface Σ_a at the point $\bar{A}_4$ meet the ideal plane at the line $\bar{A}_2\bar{P}$ and $\bar{P}$ is a linear combination of the points $\bar{A}_1, \bar{A}_3$.

Therefore, from this construction, the ray $\bar{A}_4\bar{A}_3$ coincident with the ray of the line complex which is determined from the system of equations (3).

Since the point $\bar{A}_4$ describes the surface Σ_a with tangent plane $\bar{A}_4\bar{A}_2\bar{P}$, then $d\bar{A}_4 = m\bar{P}+\omega^2{}_4\bar{A}_2$. Comparing the differential $d\bar{A}_4$ with the equation (1), it follows that, $\omega^1{}_4 \wedge \omega^3{}_4 = 0$, i.e., $\omega^3{}_4$ is a function of $\omega^1{}_4$ only. From this result and the system of equations (3), we have $\alpha = q = 0$, which characterize the constructed line complex of the subclass ${}^a(K)_I$.

2. The subclass of line complexes ${}^b(K)_I$

Any line complex $K \epsilon\, {}^b(K)_I$ is characterized by the system of differential equations (3) with the condition $\alpha = p = 0$, and the system of differential equations

$$d\begin{bmatrix}\beta\\ \gamma\\ q\\ r\end{bmatrix} = \begin{bmatrix}0 & 0 & \lambda_6 & 0\\ 0 & \mu_5 & \mu_6 & 0\\ 0 & \mu_4 & \mu_5 & 0\\ \lambda_6 & \mu_6+\beta q\nu_6 & \nu_6 & 0\end{bmatrix}\begin{bmatrix}\omega^2{}_4\\ \omega^1{}_3\\ \omega^2{}_3\\ 0\end{bmatrix} \tag{7}$$

in the 3*rd* differential neighbourhood of the ray. For the line complex $K \epsilon\, {}^b(K)_I$, $\bar{A}_3$, the centre of the ray, is a centre of inflection. From the equations (3) and (7), it is easy to prove the following existence theorem by similar way as that mentioned before.

THEOREM (2.1) : *The range of existence of the complexes belonging to the subclass ${}^b(K)_1$ comprises one arbitrary function of two arguments.*

113619

Geometrical properties of the subclass ${}^b(K)_I$:

Property (2.1) : Every line complex $K\,\epsilon\,{}^b(K)_I$ admitting a fibration into one parameter family of a hyperbolic holonomic congruence $\omega^2{}_3 = 0$, with developable surface and ideal line $\bar{A}_3\bar{A}_1$ as focal surfaces described by the distinct focal points $\bar{A}_4$ and $\bar{A}_3$ respectively.

Property (2.2) : If the ray $\bar{A}_4\bar{A}_3$ generates the line complex $K\,\epsilon\,{}^b(K)_I$, the ray $\bar{A}_4\bar{A}_2$ generates a hyperbolic line congruence C. The focal surfaces of C are the invariant line $\bar{A}_2\bar{A}_1$ and arbitrary surface with tangent plane $\bar{A}_4\bar{A}_2\bar{A}_3$ which are described by the focal points $\bar{A}_2$ and $\bar{A}_4-(k/\beta)\bar{A}_2$ respectively.

Proporty (2.3) : If the centre of the ray, $\bar{A}_3$, is stable, then the centre of the ray, $\bar{A}_4$, moves along the generator $\bar{A}_4\bar{A}_2$ of the congruence C. I.e., the ray $\bar{A}_4\bar{A}_3$ generates a bundle of lines with vertex $\bar{A}_3$ on the ideal plane and $\bar{A}_4\bar{A}_2$ as a layer line.

The integral-free representation of the subclass ${}^b(K)_I$:

From the foregoing properties of the line complex $K\,\epsilon\,{}^b(K)_I$, we have the theorem which gives the integral-free representation of this subclass.

THEOREM (2.2) : *We consider an arbitrary surface* Σ_b *and the invariant line on the ideal plane as the focal surfaces of a line congruence* C. *For each generator* m *of the line congruence* C, *construct a bundle of lines with layer on* m *and vertex on the ideal plane. All these bundles represent a line complex of the subclass* ${}^b(K)_I$.

Proof : On a generator m of the line congruence C, we place the vertices $\bar{A}_4$ and $\bar{A}_2$ of a canonical coordinate tetrahedron T $(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, the generator m intersects the ideal plane at the point $\bar{A}_2$, also the vertices $\bar{A}_1$ and $\bar{A}_3$ placed on the ideal plane and the focal points of the congruence coincident with $F = \bar{A}_4+\tau\bar{A}_2$, $\bar{A}_2$.

Since every line complex $K\,\epsilon\,{}^b(K)_I$ will be constructed as the family of all bundles of lines with vertices on the ideal plane and layers coincident with the generators of the line congruence C, hence the ray $\bar{A}_4\bar{A}_3$ generates the line complex $K\,\epsilon\,{}^b(K)_I$. Any arbitrary line complex immersed in F_3 characterized by the system of equations (3).

The point $\bar{F} = \bar{A}_4+\tau\bar{A}_2$ is a focal point of the congruence C if $d\bar{F} \equiv 0 \pmod{\bar{F}}$, which gives $\omega^1{}_4+\tau\omega^1{}_2 = 0$, $(1/\tau)\omega^3{}_4 = 0$. The value $\tau = \infty$ corresponds to the focal point $\bar{A}_2$. For the other value of τ, the forms $\omega^1{}_4$, $\omega^1{}_2$, must be satisfy $\omega^1{}_4 \wedge \omega^1{}_2 = 0$, i.e., $\omega^1{}_2$ is a function of $\omega^1{}_4$ only. From this result and using the system of equations (3) we have $\alpha = p = 0$, which characterized the constructed line complexes of the subclass ${}^b(K)_I$.

References

Finikov, S. P. (1948) : Cartan's methods of exterior forms and their applications in differential geometry Moscow.

Kovansov, N. I. (1963) : Theory of line complexes. Kiev.

Kovansov, N. I. : Ponomarov, V. G. (1975) Integral-free representation of certain complexes with a triple inflection centre on each ray in P_3.
Visnik, Kiiv. Univ. Ser. Mat. Meh. No. 17 170-175 183.

Orehova, N. S. (1970) : Complexes with a skew-symmetric matrix, Tul. Gos. Ped. Inst. Uven. Zap. Mat. Kaf. Vyp. 2 Geometr. i Algebra 41

Penner, I. A. (1967) : The differential geometry of surfaces in the space ϕ_3. Moskov. Gos. Ped. Inst. *Ucen. Zap*. No. 271, 113.

Redei, L. (1968) : Foundation of Euclidean and non-Euclidean geometries. Huingary.

Soliman, M. A. : Abdel—All, N. H. (1980) Class of line complexes in 2nd differential neighbourhood of the ray in the flag space F_3. *Bulletin of the Calcutta Math. Society*,

Veselova, O. M. (1975) : Pseudofocal transformations of holonomic nets in E_3. Kaliningrad. Gos. Univ. Differential naja Geon. Mnogoobraz. Figur No. 626-

DEPT. OF MATHS, FACULTY OF SCIENCE,
ASSIUT UNIVERSITY,
ASSIUT, EGYPT

Bull. Cal. Math. Soc., 75, 23-25 (1983)

A PARALLEL TO A GENERALIZATION OF THE MULTINOMIAL FORMULA

Aboulhassan M. Khidr

(*Received* 20 *April* 1981)

Summary **:** The value of the sum

$$\sum_{l_j \geqslant 0,\, \Sigma l_j = n} \prod_{j=1}^{m} \binom{2l_j+x_j}{l_j+p_j}$$

is obtained for particular values of x_j, and p_j, $j = 1, 2, \ldots, m$.

1. Introduction : The extension of the multinomial formula due to Hurwitz (1902), see also Riordan (1979) page (23), leads to evaluation of the sum, (All sums will be over $l_j \geqslant 0$, $j = 1, 2, \ldots, m$, and $\sum\limits_{j=1}^{m} l = n$, whenever the range is not indicated)

$$A_n(x_1, x_2, \ldots, x_m; p_1, p_2, \ldots, p_m) = \Sigma \prod_{j=1}^{m} \frac{(l_j+x_j)^{l_j+p_j}}{l_j\,!} \tag{1.1}$$

for particular values of x_j, and p_j, $j = 1, 2, \ldots, m$. These extensions are in turn the extensions of the binomial abel identities given in his (Oeuveres Completes, Christiana, C. Groendahl, 1939). In this paper we evaluate the sum

$$B_n(x_1, x_2, \ldots, x_m; p_1, p_2, \ldots, p_m) = \Sigma \prod_{j=1}^{m} \binom{2l_j+x_j}{l_j+p_j} \tag{1.2}$$

The following two sums will be parallel

$$A_n(0, 0, \ldots, 0; 0, 0, \ldots, 0) = \Sigma \prod_{j=1}^{m} \frac{l_j^{l_j}}{l_j\,!} \tag{1.3}$$

$$B_n(0, 0, \ldots, 0;\ 0, 0, \ldots, 0) = \Sigma \prod_{j=1}^{m} \binom{2l_j}{l_j} \tag{1.4}$$

The value of the sum (1.3) is found in Riordan (1979) page 25, to be equal to

$$\sum_{j=0}^{n} \binom{m-2+j}{j} \frac{n^{n-j}}{(n-j)\,!} \tag{1.5}$$

We will show in Section 2 that the sum (1.4) is equal to

$$\sum_{j=0}^{n} \binom{m-2+j}{j} \binom{2n+m-1}{n-j} \tag{1.6}$$

and then evaluate the sum (1.2) for particular values of x_j, and p_j, $j = 1, 2, \ldots, m$. In what follows the notation $C(x^n; .)$ will stand for the coefficient of x^n in the expression of the argument.

2. Proof : We first prove the following lemma.

LEMMA 2.1. *For* $0 < |4x| < 1$, *let*

$$g(x; k) = (1-4x)^{-k/2}, \text{ and } h(n, x; k) = (1-x)^{-(k-1)}(1+x)^{2n+k-1},$$

then,

$$C(x^n; g(x; k)) = C(x^n; h(n, x; k)) = \sum_{j=0}^{n} \binom{k-2+j}{j} \binom{2n+m-1}{m-j} \tag{2.1}$$

$k = 1, 2, \ldots, \quad n = 0, 1, \ldots .$

Proof of Lemma 2.1 : We proceed by induction on k. For $k = 1$, we have

$$C(x^n; g(x; 1) = \binom{2n}{n} = C(x^n; h(n, x; 1),$$

on the other hand

$$\sum_{j=0}^{n} \binom{-1}{j} \binom{2n}{n-j} = \binom{2n}{n}, \text{ since } \binom{-1}{j} = 1 \text{ for } j = 0 \text{ only.}$$

Now assume lemma is true for $k = m-1$ and write $g(x; m) = g(x; m-1)$. $g(x; 1)$, then

$$C(x^n; g(x; m)) = \sum_{j=0}^{n} C(x^j; g(x, 1)).C(x^{n-j}; g(x; m-1))$$

$$= \sum_{j=0}^{n} \binom{2j}{j} \sum_{j=0}^{n} \binom{m-3+s}{s} \binom{2n-2j+m-2}{n-j-s} \tag{2.2}$$

It can be seen that the sum in (2.2) is the coefficient of x^n in the expression

$$\sum_{w=0}^{\infty} \binom{2w}{w} x^w (1-x)^{-(m-2)}(1+x)^{2n-2x+m-2} = \{1-4x(1+x)^{-2}\}^{-\frac{1}{2}}(1-x)^{-(m-2)}(1+x)^{2n+m-2}$$

$$= (1-x)^{-(m-1)}(1+x)^{2n+m-1}$$

$$= h(n, x; m).$$

It follows that lemma is true for $k = 1, 2, \ldots .$

The proof of (1.6) follows from lemma 2.1 by noting that

$$B_n(0, 0, \ldots, 0;\ 0, 0, \ldots, 0) = \Sigma \prod_{j=1}^{m} \binom{2l_j}{l_j}$$

is equal to $C(x^n; g(x; m))$.

We are now ready to consider the sum (1.2), we need the following lemma to be able to evaluate such sum.

Lemma 2.2. *For* $0 < |4x| < 1$, *let*

$$f(x; l, m) = (1-4x)^{-l/2}\{(1-4x)^{-\frac{1}{2}}-1\}^m, \; l, m = 0, 1, \ldots,$$

and $(h(n, x; k)$ *as in lemma* 2.1, *then*

$$C(x^n; f(x; l, m) = C(x^n; h(n, x; l).(2x)^m(1-x)^{-m})$$

$$= 2^m \sum_{j=0}^{n-m} \binom{m+l-2+s}{s} \binom{2n+l-1}{n-m-s}. \tag{2.3}$$

The proof of lemma 2.2 follows of that of lemma 2.1, and therefore omitted. We illustrate by an example the use of (2.3) in evaluating the sum (1.2), consider the sum

$$B_n(1, 1, \ldots, 1; 1, 1, \ldots, 1) = B_n(1, 1, \ldots, 1. 0, 0, \ldots, 0)$$

$$= \Sigma \prod_{j=1}^{m} \binom{2l_j+1}{l_j+1} = 2^{-m} \; \Sigma \; \prod_{j=1}^{m} \binom{2l_j+2}{l_j+1}$$

$$= 2^{-m} B_n(2, 2, \ldots, 2; 1, 1, \ldots, 1)$$

$$= 2^{-m} C(x^n; f(x, 0, m)$$

$$= \sum_{j=0}^{n-m} \binom{m-2+j}{j} \binom{2n-1}{n-m-j}$$

Other possible cases can be treated similarly.

References

Hurwitz, A. (1902): Uber Abel's Verallgemeinerungen der binomischen Formel, *Acta Math.* **26,** 199 Werke II LXXXIII, 572

Riordan, J. (1979): Combinatorial Identities, R. E. Krieger Pub. Co., New York.

MATHEMATICS DEPARTMENT,
KING ABDULAZIZ UNIVERSITY,
FACULTY OF SCIENCE,
JEDDAH, SAUDI ARABIA

Bull. Cal. Math. Soc., **75**, 27–39 (1983)

GENERATION OF INTERNAL GRAVITY WAVES IN A STRATIFIED OCEAN

A. K. Pramanik

(*Received* 4 *December* 1978)

Abstract. The two-dimensional problem of gravity waves generated by a moving oscillatory pressure distribution in a two-layered fluid system is studied in details and a number of interesting conclusions concerning the wave-generation by a moving oscillatory pressure distribution is obtained. When the upper fluid is infinite in height the resulting wave pattern does not differ much from the case of homogeneous fluid but if the upper fluid is finite the wave pattern changes in a remarkable way. It is concluded that each surface of discontinuity contributes to the resulting waves one set of four waves and one critical speed and their contribution is distinctively determined. Apart from this result, a number of cases arising in concrete physical problems are considered and their effects on the waves are discussed.

1. Introduction : The two-dimensional problem of water waves generated by certain moving oscillatory pressure distribution or by oscillatory pressure distribution placed in a uniform stream has been studied under various physical conditions. Thus the problem in a homogeneous fluid has been studied by Kaplan (1957) in infinite depth, by Debnath and Rosenblat (1969) in finite uniform depth and by Pramanik (1973) in a fluid bounded by a sloping beach. The problem in a stratified fluid consisting of two layer has been studied by Pramanik (1974). The main features of the solutions may be stated as follows.

In each case the motion ultimately reduces to a steady state. The steady state comprises a set of progressive waves on the downstream side together with a set of progressive waves on the upstream side. In a homogeneous fluid which is unbounded in both the horizontal directions, the steady state wave system consists a set of four progressive waves. There exists a critical value V^* of the stream velocity or the velocity of the pressure strip V such that for $V < V^*$ the steady state consists of four waves three in the downstream sides and one in the upstream side, while for $V > V^*$, it consists of only two waves both in the downstream side and when V assumes the value V^* the solution becomes singular. In a fluid bounded by a sloping beach though the critical speed is the same as in infinite depth with no breach, the total number of ultimate waves and their distribution are slightly different. Strikingly different situation is found in a two-layered fluid. In the two-layered fluid in which the upper layer is of finite uniform depth and the lower-layer is of infinite depth and the waves are generated by the pressure distribution on the free surface, the ultimate steady state consists of two distinct sets of waves each set consisting of four progressive waves and there exists two critical speeds. Depending upon the relative magnitudes of the critical speeds and the speed of the pressure segment, the ultimate total number of waves may be eight, six or four.

This peculiar feature of waves in a two-layered fluid naturally raises the following question. Do the two distinct sets of waves and two critical speeds exist in all two-layered fluids ? This question and also the important practical applications they do have in oceanographic problems and in problems of naval hydrodynamics have aroused our interests to investigate the wave generation in two-layered fluid in more details. The present investigation is concerned with the waves in a two-layered fluid generated by a moving oscillatory pressure distribution acting on the interface of the two fluids. We at first formulate the most general problem in a fluid consisting of two finite inviscid incompressible immiscible layers of thicknesses h_1 and h_2 and densities ρ_1 and ρ_2, where the upper fluid with a free surface is of depth h_1 and density ρ_1 and the lower fluid with a rigid horizontal bottom boundary is of depth h_2 and density ρ_2, $(\rho_2 > \rho_1)$. Then we have studied in details the following three cases, (i) both the fluids are infinitely deep, i.e., both h_1 and h_2 are infinite, (ii) the upper fluid is infinite, the lower fluid is finite in depth, i.e., h_1 is infinite and h_2 is finite, and (iii) the upper fluid is finite and the lower fluid is finite, i.e., h_2 is infinite and h_1 is finite.

It is found that in the first and second cases the ultimate wave system consists of four progressive waves and one critical speed and also their distribution with respect to the critical speed is similar to the homogeneous case although the phase velocities and amplitudes are greatly modified by the stratification. In the third case the steady state wave system both on the surface and one the interface consists of two distinct sets of waves, each set containing four waves and there exist two critical speeds. A comparison of the results of these three cases and also comparison with the results of the previous cases mentioned above lead us to conclude that the steady state wave system in a two-layered fluid consists of two distinct sets of waves and two critical speeds only when there is a free surface, otherwise like in a homogeneous fluid it is only one set of four progressive waves and one critical speed although the phases and amplitude of the waves are modified to a large extent. Among the two sets of waves in the third case one is interpreted as the contribution of the free surface and other is the contribution of the interface. Besides these results, a variety of cases arising in concrete physical problems are considered and their effects on the waves are discussed.

2. Formulation. We consider a two-layered system in which the upper layer is of uniform density ρ_1 and of finite depth h_1 and the lower fluid is of uniform density ρ_2 and of finite depth h_2. For two-dimensional motion in the $x-y$ plane we consider a coordinate system in which $x-z$ plane coincides with the undisturbed horizontal interface and y-axis is positive upwards. Initially the system is at rest and the waves are produced by the forward motion with uniform velocity V of a time-dependent pressure $f(x)\exp(i\omega t)$ along the positive x-axis direction. We pose the problem in a moving coordinate system in which the origin translates with the velocity V along the positives x-axis direction so that the pressure segment is fixed in this coordinate system. Let ϕ_1 and ϕ_2 be the velocity potentials in the upper fluid and lower fluid respectively and η_1 and η_2 be respectively the free surface and interfacial elevations. Then the linearised equations of motion and the boundary conditions are

$$\nabla^2\phi_1 = 0\,, \quad 0 < y < h_1, \tag{1}$$

$$\nabla^2\phi_2 = 0, \quad -h_2 < y < 0, \tag{2}$$

$$\left.\begin{aligned} &\frac{\partial\eta_1}{\partial t} - V\,\frac{\partial\eta_1}{\partial x} = \frac{\partial\phi_1}{\partial y}, && (3)\\ &\frac{\partial\phi_1}{\partial t} - V\,\frac{\partial\phi_1}{\partial x} + g\eta_1 = 0, && (4)\end{aligned}\right\} \; y = h_1,$$

$$\left.\begin{aligned} &\frac{\partial\eta_2}{\partial t} - V\,\frac{\partial\eta_2}{\partial x} = \frac{\partial\phi_2}{\partial y} = \frac{\partial\phi_1}{\partial y}, && (5)\\ &s\left(\frac{\partial\phi_1}{\partial t} - V\,\frac{\partial\phi_1}{\partial x} + g\eta_2\right) \\ &-\left(\frac{\partial\phi_2}{\partial t} - V\,\frac{\partial\phi_2}{\partial x} + g\eta_2\right) = f'(x)e^{i\omega t}, && (6)\end{aligned}\right\} \; y = 0,$$

$$\frac{\partial\phi_2}{\partial y} = 0, \quad y = -h_2, \tag{7}$$

where,

$$f'(x) = \frac{1}{\rho_2}\,f(x), \frac{\rho_1}{\rho_2} = s.$$

The initial conditions give

$$\phi_1 = \phi_2 = 0,\ \eta_1 = \eta_2 = 0, \text{ at } t = 0. \tag{8}$$

3. Formal Solution. In order to solve the problem posed above we introduce Fourier transform with respect to x defined as

$$\bar{\varphi}(k, y, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x, y, t) e^{-ikx} dx.$$

The solutions for the functions $\overline{\varphi_1}$ and $\overline{\varphi_2}$ are obtained from the transformed equations (1) and (2) with the help of the transformed condition (7) in the form

$$\overline{\varphi_1} = A \cosh|k|y + B \sinh|k|y, \tag{9}$$

$$\overline{\varphi_2} = C \cosh|k|(y + h_2), \tag{10}$$

where A, B and C are functions of k and t.

The interfacial condition (5) gives the following relation between B and C,

$$B = C \sinh|k|h_2. \tag{11}$$

Eliminating η_1 from (3) and (4) and η_2 from (5) and (6) and then transforming the resulting equations on φ_1 and ϕ_2 respectively and utilising (9), (10) and (11) we obtain

$$\left.\begin{aligned} &(D^2A+g|k|B)\cosh|k|h_1+(D^2B+g|k|A)\sinh|k|h_1 = 0,\\ &s(D^2A+g|k|B)-D^2B\coth|k|h_2+g|k|B) = i(\omega-kV)\bar{f}(k)e^{i\omega t}, \end{aligned}\right\} \tag{12}$$

where $\bar{f}(k)$ is the Fourier transform of $f'(x)$ and $D = \dfrac{\partial}{\partial t}-ikV$. Elimination of B gives the following equation for A,

$$\begin{aligned} &\frac{\partial^4 A}{\partial t^4}-4ikV\frac{\partial^3 A}{\partial t^3}+\left\{-6k^2V^2+\frac{g|k|(\coth|k|h_1+\coth|k|h_2)}{s+\coth|k|h_1\coth|k|h_2}\right\}\frac{\partial^2 A}{\partial t^2}\\ &+2ikV\left\{2k^2V^2-\frac{g|k|(\coth|k|h_1+\coth|k|h_2)}{s+\coth|k|h_1\coth|k|h_2}\right\}\frac{\partial A}{\partial t}+\left\{k^4V^4\right.\\ &\left.-\frac{g|k|k^2V^2(\coth|k|h_1+\coth|k|h_2)}{s+\coth|k|h_1\coth|k|h_2}+\frac{g^2|k|^2(1-s)}{s+\coth|k|h_1\coth|k|h_2}\right\}A\\ &=\frac{i(\omega-kV)\bar{f}(k)[g|k|\coth|k|h_1-(\omega-kV)^2]}{s+\coth|k|h_1+\coth|k|h_2}e^{i\omega t}. \end{aligned} \tag{13}$$

Solution to this differential equation gives

$$A = A_1e^{i(kV+\sigma_1)t}+A_2e^{i(kV-\sigma_1)t}+A_3e^{i(kV+\sigma_2)t}+A_4e^{i(kV-\sigma_2)t}+Pe^{i\omega t} \tag{14}$$

where,

$$\begin{aligned} \sigma_1, \sigma_2 = {}&\frac{g|k|}{2(1+s\tanh|k|h_1+\tanh|k|h_2)}](\tanh|k|h_1+\tanh|k|h_2)\pm\{(\tanh|k|h_1\\ &+\tanh|k|h_2)^2-4(1-s)\tanh|k|h_1\tanh|k|h_2(1+s\tanh|k|h_1\tanh|k|h_2)\}^{\frac{1}{2}}] \end{aligned}$$

$$P = \frac{i\bar{f}(k)(\omega-kV)\tanh|k|h_2[g|k|-(\omega-kV)^2\tanh|k|h_1].}{(1+s\tanh|k|h_1\tanh|k|h_2)[(\omega-kV)^2-\sigma_1{}^2][(\omega-kV)^2-\sigma_2{}^2]} \tag{15}$$

The following values of the four functions A_j, $j = 1, 2, 3, 4$ are determined from the initial conditions,

$$A_1 = A(1, 2, 1),\ A_2 = A(1, 2, -1),\ A_3 = -A(2, 1, 1),\ A_4 = -A(2, 1, -1),$$

where

$$2\sigma_r(\sigma_1^2-\sigma_2^2)A(\gamma,\mu,\lambda_1)=[\sigma_\mu^2-(\omega-kV)^2][\sigma_r+\lambda(\omega-kV)]P$$

$$+\lambda[\sigma_\mu^2-(\omega-kV)(\lambda\sigma_r+\omega-kV)]\;\frac{i\bar{f}(k)\tanh|k|h_1\tanh|k|h_2}{1+s\tanh|k|h_1\tanh|k|h_2}$$

$$-\lambda[\tanh^2|k|h_1-1-(1-s)\tanh|k|h_1\tanh|k|h_2]\frac{i\bar{f}(k)g|k|\tanh|k|h_2}{1+s\tanh|k|h_1\tanh|k|h_2}.$$

The functions $\bar{\eta}_1$ and $\bar{\eta}_2$ can be obtained from the transformed equations (4) and (6) on utilising the equations (9), (10) and (14). It is found that

$$\begin{aligned}g\bar{\eta}_1 &= i\sigma_1 e^{ikVt}(-A_1e^{i\sigma_1t}+A_2e^{-i\sigma_1t})\sinh|k|h_1\mathrm{E}_1(\sigma_1)\\ &+i\sigma_2e^{ikVt}(-A_3e^{i\sigma_2t}+A_4e^{-i\sigma_2t})\sinh|k|h_1\mathrm{E}_1(\sigma_2)\\ &+iP(kV-\omega)e^{i\omega t}\mathrm{E}_1(kV-\omega)\\ &+\frac{\bar{f}(k)(kV-\omega)^2\sinh|k|h_1\tanh|k|h_1\tanh|k|h_2}{g|k|[1+(s-1)\tanh|k|h_1\tanh|k|h_2]},\end{aligned}\tag{16}$$

$$\begin{aligned}g(s-1)\bar{\eta}_2 &= e^{ikVt}(A_1e^{i\sigma_1t}-A_2e^{-i\sigma_1t})\mathrm{E}_2(\sigma_1)\\ &+e^{ikVt}(A_3e^{i\sigma_2t}-A_4e^{-t\sigma_2t})\mathrm{E}_2(\sigma_2)\\ &+\rho e^{t\omega t}\mathrm{E}_2(kV-\omega)\\ &+\bar{f}(k)e^{i\omega t}\left\{1-\frac{(\omega-kV)^2\tanh|k|h_1}{g|k|[1+(s-1)\tanh|k|h_1\tanh|k|h_2],}\right\}\end{aligned}\tag{17}$$

where

$$\mathrm{E}_1(x)=ix\sinh|k|h_1\left\{\coth|k|h_1+\frac{(1+s\tanh|k|h_1\tanh|k|h_2)x^2-g|k|\tanh|k|h_1}{g|k|[1+(s-1)\tanh|k|h_1\tanh|k|h_2]}\right\}$$

and

$$\mathrm{E}_2(x)=ix\left\{\frac{(1+s\tanh|k|h_1\tanh|k|h_2)x^2-g|k|\tanh|k|h_2}{g|k|\tanh|k|h_2[1+(s-1)\tanh|k|h_1\tanh|k|h_2]}-1\right\}.\tag{18}$$

The function $\bar{\eta}_1$ and $\bar{\eta}_2$ being determined the free surface displacement η_1 and interfacial displacement η_2 are then determined in the integral form by Fourier inversion formula.

4. Steady State Solution : The steady state waves will be determined by proceeding to the limit as time goes to infinity in the integral representations for η_1 and η_2. It is easy to see that the integrals representing η_1 and η_2 have the identical features as those in the previous cases (Debnath and Rosenblat, 1969 and Pramanik, 1973). Thus as before the integrals for η_1 and η_2 can each be put as sum of several integrals each of which contains one or two poles. And thus the component integrals can be evaluated asymptotically by the same method as done there (Debnath and Rosenblat, 1969 and Pramanik, 1973). We shall not give here the details of these asymptotic calculations which can be easily done by the previous method. We shall only write down the results. As mentioned in the introduction the subsequent analysis is concerned with the three cases (i), (ii) and (iii). In the following we shall denote the desired functions by a two-suffixed notation η_{ij} the first suffix will denote the case and the second will denote the free surface or interface. For example, η_{32} denotes the interfacial displacement in third case.

4.1. *Steady State value of* η_{12}. The integral representation for the interfacial elevation can be obtained from the previous expressions by proceeding to the limit as h_1 and h_2 both go to infinity. Hence making $h_1 \to \infty$ and $h_2 \to \infty$, we obtain

$$\sqrt{2\pi} g(1-s)\eta_{12} = J(k)+J(-k),$$

$$\begin{aligned}
J(k) = &\int_0^\infty \frac{\bar{f}(k)(1+s)(\omega-kV)[\sigma_1{}^2-(\omega-kV)^2]}{4s\sigma_1{}^2[\sigma_2-(\omega-kV)]}\, e^{i[(kV+\sigma_2)t+kx]}dk \qquad (19)\\
&-\int_0^\infty \frac{\bar{f}(k)(1+s)(\omega-kV)[\sigma_1{}^2-(\omega-kV)^2]}{4s\sigma_1{}^2[\sigma_2+(\omega-kV)]}\, e^{i[(kV-\sigma_2)t+kx]}dk\\
&+\int_0^\infty \frac{\bar{f}(k)[2s\sigma_1{}^2+(\omega-kV)(kV+\sigma_2-\omega)(s+1)]}{2(1+s)(\sigma_1{}^2-\sigma_2{}^2)}\, e^{i[(kV+\sigma_2)t+kx]}dk\\
&+\int_0^\infty \frac{\bar{f}(k)[2s\sigma_1{}^2+(\omega-kV)(kV-\sigma_2-\omega)(s+1)]}{2(1+s)(\sigma_1{}^2-\sigma_2{}^2)}\, e^{i[(kV-\sigma_2)t+kx]}dk\\
&+e^{i\omega t}\int_0^\infty \frac{\bar{f}(k)(\omega-kV)^2[(1+s)(\omega-kV)^2-(1+s^2)\sigma_1{}^2]}{s(1+s)\sigma_1{}^2[\sigma_2{}^2-(\omega-kV)^2]}\, e^{ikx}dk\\
&-e^{i\omega t}\int_0^\infty \frac{\bar{f}(k)[\sigma_1{}^2 s-(\omega-kV)^2]}{s\sigma_1{}^2}\, e^{ikx}dk,
\end{aligned}$$

where

$$\sigma_1{}^2 = gk, \quad \sigma_2{}^2 = \frac{1-s}{1+s}\, gk.$$

As mentioned above η_{12} has been expressed into sum of several integrals each of which contains either no pole, or one or two of the following poles,

$$\alpha_1{}^{\frac{1}{2}}, \alpha_2{}^{\frac{1}{2}} = \frac{1}{2V}\left[\left(\frac{1-s}{1+s}\,g\right)^{\frac{1}{2}} \pm \left(\frac{1-s}{1+s}\,g-4\omega V\right)^{\frac{1}{2}}\right], \text{ for } \frac{1-s}{1+s}\,g > 4\omega V,$$

$$\alpha_4{}^{\frac{1}{2}}, \alpha_3{}^{\frac{1}{2}} = \frac{1}{2V}\left[\pm\left(\frac{1-s}{1+s}\,g\right)^{\frac{1}{2}}+\left(\frac{1-s}{1+s}g+4\omega V\right)^{\frac{1}{2}}\right].$$

The roots α_1 and α_2 become coincident when

$$4\omega V = \frac{1-s}{1+s}\, g.$$

We designate this case by writing $V = V_1^*$. Then by the method mentioned above the following steady state value of η_{12} can be easily obtained

$$\left.\begin{aligned}\sqrt{2\pi}g(1-s)\eta_{12} &= \bar{f}(\alpha_3)P_1(\alpha_3)e^{i(\omega t+\alpha_3 x)}+\bar{f}(\alpha_4)F_1(\alpha_4)e^{i(\omega t+\alpha_4 x)} \\ &\quad +\bar{f}(-\alpha_1)F_1(\alpha_1)H(V_1^*-V)e^{i(\omega t-\alpha_1 x)},\ x\to-\infty \\ &= -\bar{f}(-\alpha_2)F_1(\alpha_2)H(V_1^*-V)e^{i(\omega t-\alpha_2 x)},\ x\to\infty\end{aligned}\right\} \tag{20}$$

where H denotes Heaviside step function and

$$P_1(k) = \frac{\sigma_2}{\sigma_2'+V},\quad F_1(k) = \frac{\sigma_2}{\sigma_2'-V};$$

σ_2' denoting differentiation with respect to k.

4.2. *Steady State Value of* η_{22}. The interfacial displacement η_{22} of the present case can be easily obtained by proceeding to the limit as $h_2\to\infty$. We obtain

$$\sqrt{2\pi}g(1-s)\eta_{22} = J(k)+J(-k),$$

$$\begin{aligned}J(k) = &-e^{i\omega t}\int_0^\infty\left[1-\frac{(\omega-kV)^2}{\sigma_1^2\{1+(s-1)\tanh kh\}}\right]e^{ikx}\,dk \\ &+\int_0^\infty\frac{\bar{f}(k)(\omega-kV)(1+s\tanh kh_2)[\sigma_1^2-(\omega-kV)^2]}{2\sigma_1^2[\sigma_2-(\omega-kV)][1-\tanh kh_2+2s\tanh kh_2]}\,e^{i[kx+(kV+\sigma_2)t]}\,dk \\ &-\int_0^\infty\frac{\bar{f}(k)(\omega-kV)(1+s\tanh kh_2)[\sigma_1^2-(\omega-kV)^2]}{2\sigma_1^2[\sigma_2+(\omega-kV)][1-\tanh kh_2+2s\tanh kh_2]}\,e^{i[kx+(kV-\sigma_3)t]}\,dk. \\ &+e^{i\omega t}\int_0^\infty\frac{\bar{f}(k)(\omega-kV)^2[(1+s\tanh kh_2)(\omega-kV)^2-\sigma_1^2\{1+s^2+s(1-\tanh kh_2)\}]}{\sigma_1^2(1+s\tanh kh_2)[1+(s-1)\tanh kh_2][\sigma_2^2-(\omega-kV)^2]}\,e^{ikx}\,dk \\ &+\int_0^\infty\frac{\bar{f}(k)[\sigma_1^2-\sigma_2^2+(\omega-kV)(kV-\sigma_2-\omega)]}{2(\sigma_1^2-\sigma_2^2)}\,e^{i[kx+(kV+\sigma_2)t]}\,dk \\ &+\int_0^\infty\frac{\bar{f}(k)[\sigma_1^2-\sigma_2^2+(\omega-kV)(kV+\sigma_2-\omega)]}{2(\sigma_1^2-\sigma_2^2)}\,e^{i[kx+(kV-\sigma_2)t]}\,dk,\end{aligned}$$

where,
$$\sigma_1^2 = gk,\quad \sigma_2^2 = \frac{(1-s)gk\tanh kh_2}{1+s\tanh kh_2}$$

A scrutiny of the integrals reveals that the poles of the integrands are the solution of the following equations

$$\sigma_2 - kV + \omega = 0,\ \sigma_2 - kV - \omega = 0,\ \sigma_2 + kV - \omega = 0.$$

The roots of these equations have been determined by Pramanik (1974) as the points of intersection of the curve σ_2 with the straight line $kV-\omega$, $kV+\omega$ and $-kV+\omega$. For our present purpose we state the results. In general there are four roots k_n, $n = 1, 2, 3, 4$ of the system. For certain critical value $V_2{}^*$ of V the roots k_1 and k_2 become coincident. This case can be represented by a curve AB (Figure I) in the $(a-b)$ plane where a and b are defined by

$$a = \frac{V\omega}{g},\ b = \frac{V}{\sqrt{gh}}. \tag{21}$$

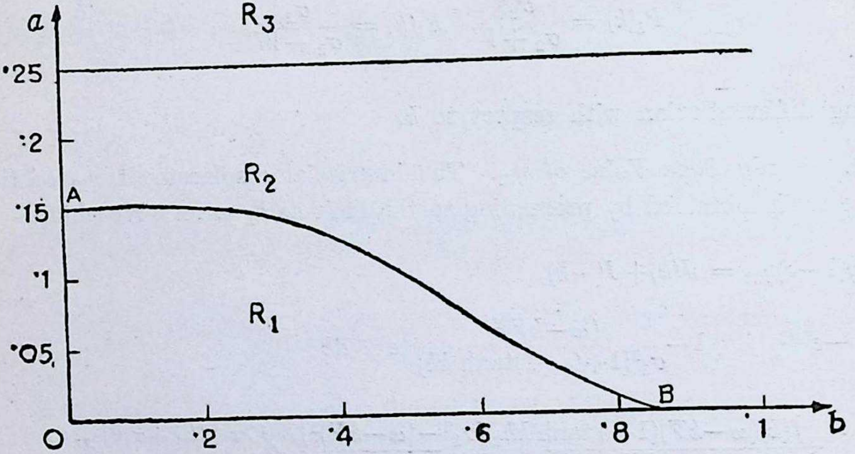


FIG. 2. Critical curves in the (a- b) plane.

Then it can be readily verified that if the values of the parameters, V, h_2, ω are such that the point (a, b) lies below the curve AB then all the roots are present, while if (a, b) lies above AB then only k_3 and k_4 are real. The steady state value of η_{22} can now be written down in terms of the poles k_n. In the following we write the result when the point (a, b) is below curve AB—

$$\left.\begin{aligned}\sqrt{2\pi}g(1-s)\eta_{22} &= \bar{f}(k_3)P_2(k_3)e^{i(\omega t+k_3x)} + \bar{f}(k_4)F_2(k_4)e^{i(\omega t+k_4x)}\\ &\quad + \bar{f}(-k_1)F_2(k_1)e^{i(\omega t-k_1x)},x\to-\infty,\\ &= \bar{f}(-k_2)F_2(k_2)e^{i(\omega t-k_2x)},x\to\infty,\end{aligned}\right\} \tag{22}$$

where

$$[P_2(k),\, F_2(k)] = \frac{\pi i(1+s)}{(1+s\tanh kh_2)[1+(s-1)\tanh kh_2]}\left[\frac{\sigma_2}{\sigma_2' \pm V}\right].$$

When the point (a, b) lies above the curve AB, in the expression η_{22}, the waves corresponding to the roots k_1 and k_2 will not occur.

4.3. *Steady State Values of* η_{31} *and* η_{32}. The integral representations for η_{31} and η_{32} can be easily determined from the general expressions in section 3 by proceeding to the limit as $h_2 \to \infty$. The functions η_{31} and η_{32} can be put in the following integral forms,

$$\sqrt{2\pi}g\eta_{31} = \mathcal{I}(k)+\mathcal{I}(-k),$$

where

$$\mathcal{I}(k) = e^{i\omega t}\int_0^\infty \frac{\bar{f}(k)(\omega-kV)^2[\sigma_1^2-(\omega-kV)^2\tanh kh_1]}{(1+s\tanh kh_1)[\sigma_1^2-(\omega-kV)^2][\sigma_2^2-(\omega-kV)^2]}$$

$$\times\left\{\frac{(\omega-kV)^2(1+s\tanh kh_1)-(\sigma_1^2\tanh kh_1]\sinh kh_1}{\sigma_1^2[(s-1)\tanh kh_1+1]}+\cosh kh_1\right\}e^{iku}dk$$

$$+\int_0^\infty i\sigma_1(\cosh kh_1+\sinh kh_1)(-A_1e^{i\sigma_1 t}+A_2e^{-i\sigma_1 t})e^{ik(Vt+x)}dk$$

$$+\int_0^\infty i\sigma_2\left\{1-\frac{s\tanh^2 kh_1}{(s-1)\tanh kh_1+1}\right\}(-A_3e^{i\sigma_2 t}+A_4e^{-i\sigma_2 t})e^{ik(Vt+x)}dk$$

$$+e^{i\omega t}\int_0^\infty \frac{\bar{f}(k)kV(\omega-kV)\tanh kh_1\sinh kh_1}{\sigma_1^2[(s-1)\tanh kh_1+1]}e^{ikx}dk,$$

and

$$\sqrt{2\pi}g\eta_{32} = J(k)+J(-k),$$

where

$$J(k) = e^{i\omega t}\int_0^\infty \frac{\bar{f}(k)(\omega-kV)^2[\sigma_1^2-(\omega-kV)^2\tanh^2 kh_1]}{(1-s)\sigma_1^2(1+s\tanh kh_1)[\sigma_1^2-(\omega-kV)^2[\sigma_2^2-(\omega-kV)^2]}$$

$$\times\frac{(\omega-kV)^2(1+s\tanh kh_1)-\sigma_1^2[s+(1+s^2-s)\tanh kh_1]}{(s-1)\tanh kh_1+1}e^{ikx}dk$$

$$+\int_0^\infty i\sigma_1(-A_1e^{i\sigma_1 t}+A_2e^{-i\sigma_1 t})e^{ik(Vt+x)}dk$$

$$+\int_0^\infty \frac{i\sigma_2 s(1+s\tanh kh_1)}{(1-s)[(s-1)\tanh kh_1+1]}(A_3e^{i\sigma_2 t}-A_4e^{-i\sigma_2 t})e^{ik(Vt+x)}dk$$

$$+e^{i\omega t}\int_0^\infty \frac{\bar{f}(k)}{\sigma_1^2}\,\frac{\sigma_1^2[(s-1)\tanh kh_1+1]-kV(\omega-kV)\tanh kh_1}{(s-1)\tanh kh_1+1}\,e^{ikx}dk.$$

The functions A_n in the above expressions are given by the following relations.

$$A_1 = A(1, 2, 1),\ A_2 = A(1, 2, -1),\ A_3 = -A(2, 1, 1),\ A_4 = -A(2, 1, -1),$$

where

$$2\sigma_r(\sigma_1^2-\sigma_2^2)A(\gamma, \mu, \lambda) = [\sigma_\mu^2-(\omega-kV)^2][\sigma_r+\lambda(\omega-kV)]P$$

$$+\lambda[\sigma_\mu^2-(\omega-kV)(\lambda\sigma_r+\omega-kV)]\frac{i\bar{f}(k)\tanh kh_1}{1+s\tanh kh_1}$$

$$-\lambda[(\tanh^2 kh_1-1-(1-s)\tanh kh_1]\,\frac{i\bar{f}(k)\sigma_1^2}{1+s\tanh kh_1}$$

$$\sigma_1^2 = gk,\quad \sigma_2^2 = \frac{\sigma_1^2(1-s)\tanh kh_1}{1+s\tanh kh_1}$$

$$P = \frac{i\bar{f}(k)(\omega-kV)[\sigma_1^2-(\omega-kV)^2\tanh kh_1]}{(1+s\tanh kh_1)[(\omega-kV)^2-\sigma_1^2][(\omega-kV)^2-\sigma_2^2]}$$

A scrutiny of the integrals in the expressions for η_{31} and η_{32} reveals that the poles of the integrands are the solutions of the following equations

$$\sigma_1+kV-\omega = 0,\ \sigma_1-kV-\omega = 0,\ \sigma_1-kV+\omega = 0, \tag{A}$$

$$\sigma_2+kV-\omega = 0,\ \sigma_2-kV-\omega = 0,\ \sigma_2-kV+\omega = 0. \tag{B}$$

The system (A) at once admits of the following solutions

$$\beta_1, \beta_2 = \frac{1}{2V^2}[g-2V\omega\pm\sqrt{g^2-4gV\omega}],$$

$$\beta_3, \beta_4 = \frac{1}{2V^2}[g+2V\omega\pm\sqrt{g^2+4gV\omega}].$$

The roots β_1 and β_2 become coincident if $V = g/4\omega$. We designate this case by writing $V = V^*$. The solution of the system (B) has already been mentioned in section (4.2). Following the method as in (Pramanik, 1974) the critical cases $V = V^*$ and $V = V_2^*$ can be represented by a straight line and a curve (Figure I) in the $(a-b)$ plane where a and b are defined in (21). Then the following statements regarding the existence of the roots β_n and k_n can be easily verified. The critical curves divide the whole quadrant of the $(a-b)$

plane into three regions R_n such that if the point (a, b) lies in R_1 all the roots β_n and k_n are present, when the point (a, b) is in R_2 all β_n and k_3, k_4 are present and when (a, b) lies in R_3 only β_3, β_4 and k_3, k_4 are present.

We can now write down the steady state values of η_{31} and η_{32}. It is convenient to write them as sum of two components as follows.

$$\sqrt{2\pi}\eta_{31} = \eta'_{31}+\eta''_{31},$$

$$\sqrt{2\pi}\eta_{32} = \eta'_{32}+\eta''_{32}.$$

In the following we write down the expressions for these components for the case when the point $(a\ b)$ lies in R_1.

$$\begin{aligned}\eta'_{3n} &= \bar{f}(-\beta_2)F_{n1}(\beta_2)e^{-i\beta_2 x+i\omega t}, \quad x \to \infty \\ &= -\bar{f}(-\beta_1)F_{n1}(\beta_1)e^{i\beta_1 x+i\omega t}+\bar{f}(\beta_3)F_{n1}(\beta_3)e^{i\beta_3 x+i\omega t} \\ &\quad +\bar{f}(\beta_4)P_{n1}(\beta_4)e^{i\beta_4 x+i\omega t}, \quad x \to -\infty,\end{aligned}$$

$$\begin{aligned}(-1)^n\eta''_{3n} &= \bar{f}(-k_2)F_{n2}(k_2)e^{ik_2 x+i\omega t}, \quad x \to \infty \\ &= -\bar{f}(-k_1)F_{n2}(k_1)e^{-ik_1 x+i\omega t}+\bar{f}(k_3)F_{n2}(k_3)e^{ik_3 x+i\omega t} \\ &\quad +\bar{f}(k_4)P_{n2}(k_4)e^{ik_4 x+i\omega t} \quad x \to -\infty,\end{aligned}$$

$$n = 1,\ 2.$$

where

$$F_{rt} = F(r, t, -1),\ P_{rt} = F(r, t, 1),$$

$$F(r, t, \lambda) = \frac{\pi i f_r(h_1)\sigma_t}{[1+(2s-1)\tanh kh_1](\sigma'_t+\lambda V)},$$

$$f_1(h_1) = \operatorname{sech} kh_1, \quad f_2(h_1) = 1-\tanh kh_1.$$

The waves for the cases when the point (a, b) lies in R_2 or in R_3 can be easily determined. These are simply the waves expressed above only the waves corresponding to the poles not present in any case being deleted for that case, the direction of propagation of the other waves remaining the same.

Thus we see that when the point (a, b) lies in R_1 the steady state wave system both on the free surface and on the interface contains eight progressive waves, six downstream and two upstream, when the point (a, b) lies in R_2, it contains six waves, five downstream and one upstream, and when the point (a, b) lies in R_3, it contains only four waves all downstream.

5. **Discussions of the results** A comparison between the results of the three cases at once reveals that when the upper fluid is infinite the resulting wave system contains one set of four progressive waves and one critical speed like in a homogeneous fluid, while when the upper fluid is finite, the wave system both at the free surface and on interface contains two distinct sets of waves each set containing four progressive waves and two critical speeds. We can conclude thus that the resulting wave system contains two distinct sets and two critical speeds only when there is a free surface, otherwise like in a homogeneous fluid it is one set of four waves and one critical speed with however modified phases and amplitudes. We can put this result in a different manner. We see that in the first two cases there is one surface of discontinuity while in the third case there are two. So, we can say that there are two sets of waves when there are two surfaces of discontinuity, and each surface of discontiniuity contributes to the resulting wave pattern one set of four waves and one critical speed.

The first two cases in which the upper fluid is infinite can be considered to be effect of atmosphere on the waves generated on a homogeneous ocean, while the third case can be taken as an stratified ocean having a layer of separate density over an infinite homogeneous fluid. Thus we see that the atmosphere, apart from modifying the phases and amplitudes has no other great effect on the waves generated on the ocean, while a layer of separate density on a homogeneous ocean modifies the resulting wave pattern to a great extent.

We now like to discuss the various limiting cases. The waves in a homogeneous fluid can be obtained from our results. Thus proceeding to the limit as $\rho_1 \to 0$ in the steady state expressions for η_{12} and η_{22} we find the results of Kaplan (1957) and Debnath and Rosenblat (1969). The results of Kaplan can also be obtained from η_{31} and η_{32} by proceeding to the limit as $h_1 \to \infty$. As we proceed to the limit $h_2 \to 0$, η_{22} coincides with η_{12}. The steady state expressions for η_{31} and η_{32} deserve special attention for various values of h_1. As h_1 decreases to zero, the components η''_{31} and η''_{32} decrease monotonically to zero, while η'_{31} and η'_{32} approach the same value of homogeneous case. As we increase h_1 from zero to infinitely, the component η''_{31} at first increases to a maximum and then decreases to zero, and η'_{31} and η'_{32} decrease monotonically to zero, while the component η''_{32} approaches to a value and coincides with η_{21}. We have observed earlier that each surface of discontinuity contributes to the resulting wave system one set of four waves and one critical speed. The foregoing discussion leads us to conclude that among the two components of η_{31} and η_{32} the first component η'_{31} and η'_{32} are the contributions of the free surface and the second components η''_{31} and η''_{32} are that of the interface.

Another interesting case we want to discuss is the behaviour of the waves for different values of ρ_1 and ρ_2. Here we shall discuss two limiting cases when ρ_2 is much larger than ρ_1 and when ρ_1 and ρ_2 are nearly equal. If we are interested in the effects of atmosphere on the ocean waves, then since the density of water is 770 times greater than that of air, we can put $\rho_1/\rho_2 = s$ where s is small. Then a very simple calculation reduces the amplitude of the first wave in η_{12} to

$$\frac{i\sqrt{\bar{g}}\alpha'_3}{\rho_2(\sqrt{g}-2V\sqrt{\alpha'_3})}+O(s),$$

where

$$\alpha'_3 = \frac{1}{2V^2}(g+2V\omega-\sqrt{g^2+4gV\omega}).$$

The first term represents the wave in the corresponding infinite homogeneous fluid. Similar analysis holds for other waves. This clearly shows that the effects of the upper fluid is negligible in the present case. On the other hand when ρ_1 and ρ_2 are nearly equal we can put $\rho_2 = \rho_1(1+s)$ where s is small. Then a little calculation will show that the amplitude of any wave is of the order of $s^{-1/2}$, which evidently shows that the wave amplitude becomes very large in this case.

References

Debnath, L. and Rosenblat, S. (1969) : The ultimate approach to the steady state in the generation of waves on a running stream. *Q. Jl. Mech. Appl. Math.*, 22.

Kaplan, P. (1957) : The waves generated by the forward motion of oscillatory pressure distribution. *Proc. 5th Midwest Conf. Fluid Mech.*, 316.

Pramanik, A. K. (1973) : The waves generated by a moving oscillatory surface pressure on a sloping beach. *ZAMM*. 53.

Pramanik, A. K. (1974) : Waves due to a moving oscillatory surface pressure in a stratified fluid. *Jou. Appl. Mech.* 41.

DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY COLLEGE OF SCIENCE
CALCUTTA UNIVERSITY
92, A. P. C. ROAD,
CALCUTTA-700 009, INDIA

[illegible]

[illegible]

[illegible]

[illegible] very large in this case.

References

[illegible]

[illegible]

[illegible]

[illegible]

DEPARTMENT OF APPLIED MATHEMATICS
UNIVERSITY COLLEGE OF SCIENCE
CALCUTTA UNIVERSITY
92, A. P. C. ROAD
CALCUTTA-700 009, INDIA

Bull. Cal. Math. Soc. 75, 41–47 (1983),

CLASSIFICATION OF LINE COMPLEXES IN THE IST DIFFERENTIAL NEIGHBOURHOOD OF IST RAY IN ISOTROPIC SPACE I_3

M. A. Soliman and N. H. Abdel-All

(*Received* 26 *September* 1980)

Abstract. A line complex in geometry of lines is an arbitrary 3-parameter system of lines, a line congruence is an arbitrary 2-parameter system of lines and ruled surface is an arbitrary one-parameter system of lines. The present work is concerned mainly with the classification of the line complexes and their geometrical properties in the 1st differential neighbourhood of its ray in isotropic space I_3. The method adopted here is based on Cartan's differential calculus (Finkov 1948).

1. Introduction Isotropic space I_3 is a 3-dimensional projective space P_3 with two times degenerate absolutum consisting of a plane with two conjugate intersecting lines in the sence of complex variable. In the space P_3 we use x^1, x^2, x^3, x^4 as homogeneous point coordinates. We consider the geometry belonging to the following sixfold group of projective transformations (Strubecker 1961) :

$$\begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \\ \bar{x}^3 \\ \bar{x}^4 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 & a_{14} \\ \sin\phi & \cos\phi & 0 & a_{24} \\ a_{31} & a_{32} & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

In isotropic space I_3 it is possible to take affine system of coordinates. As usual in affine geometry, we call the invariant plane $x^4=0$ the ideal plane. For the points which do not lie in that plane, we take $x^4=1$. Our geomtry has the properties of the affine and therefore also of the projective space.

The distance of the ordered pair of points $P(x^1, x^2, x^3)$ and $O(y^1, y^2, y^3)$ is $\{(y^2-x^2)^2+(y^1-x^1)^2\}^{\frac{1}{2}}$. If this distance is zero for two different points, we define $|y^3-x^3|$ as distance of second order. The angle between the ordered pair of planes $x^1+\xi_2x^2+\xi_3x^3+\xi_4x^4=0$ and $x^1+\eta_2x^2+\eta_3x^3+\eta_4x^4=0$ is dual to the distance and is given by $\xi_2-\eta_2$. If this angle is zero, we may introduce an angle of higher order.

We choose moving frame conjugate to any arbitrary manifold immersed in I_3 as coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, where the vertices $\bar{A}_1, \bar{A}_2, \bar{A}_3$ lie on the ideal plane and $\bar{A}_4$ is proper point such that $\bar{A}_4\bar{A}_1$, $\bar{A}_4\bar{A}_2$ and $\bar{A}_4\bar{A}_3$ form orthogonal traid. If we take the point of intersection of the two lines of the absolutum as the vertex $\bar{A}_3$ then the absolutum will have in the mentioned tetrahedron the equation $x^4 = 0$, $(x^1)^2+(x^2)^2 = 0$. The geometry is completely self dual (Redei 1968).

The fundamental equations of the moving frame in the space I_3 are

$$d\begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix} = \begin{bmatrix} 0 & \omega^2{}_1 & \omega^3{}_1 & 0 \\ \omega^1{}_2 & 0 & \omega^3{}_2 & 0 \\ 0 & 0 & 0 & 0 \\ \omega^1{}_4 & \omega^2{}_4 & \omega^3{}_4 & 0 \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \\ \bar{A}_4 \end{bmatrix}, \ \omega^2{}_1+\omega^1{}_2 = 0 \tag{1}$$

and the structural equations (the integrability conditions) are

$$D\begin{bmatrix} \omega^2{}_1 \\ \omega^3{}_1 \\ \omega^3{}_2 \\ \omega^1{}_4 \\ \omega^2{}_4 \\ \omega^3{}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega^2{}_1 & 0 & 0 & 0 \\ 0 & -\omega^2{}_1 & 0 & 0 & 0 & 0 \\ -\omega^2{}_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\omega^2{}_1 & 0 & 0 \\ 0 & \omega^1{}_4 & 0 & 0 & -\omega^3{}_2 & 0 \end{bmatrix} \Lambda \begin{bmatrix} \omega^2{}_1 \\ \omega^3{}_1 \\ \omega^3{}_2 \\ \omega^1{}_4 \\ \omega^2{}_4 \\ \omega^3{}_4 \end{bmatrix}, \tag{2}$$

where $\omega^j{}_i$ are Pffaf's forms, D denotes the exterior differentiation operator and Λ the exterior product between the differential forms.

2. Classification of Line Complexes in I_3.

Let, in I_3, u, v, w be a system of curvelinear coordinates; the admissible variables (u, v, w) are taken from an open neighbourhood of C^3. Consider a line complex immersed in I_3, which generated by the ray $l = (\bar{A}_2\bar{A}_4)$, $l = l(u, v, w)$. It is well known how to represent the lines of the space I_3 as points of the Grassmann manifold of dimension four in the $g(I_3)$ of dimension five; we denote by $(\bar{A}_2\bar{A}_4)$ the Grassmann coordinates of the line $l = (\bar{A}_2\bar{A}_4)$. From the differential $dl = \omega^1{}_2(\bar{A}_1\bar{A}_4)+\omega^3{}_2(\bar{A}_3\bar{A}_4)+\omega^1{}_4(\bar{A}_2\bar{A}_1)+\omega^3{}_4(\bar{A}_2\bar{A}_3)$, it follows that, the differential equation of the line complex related to the moving frame T, in the 1*st* differential neighbourhood of its ray given by

$$\omega^1{}_2 = a\omega^3{}_2+b\omega^1{}_4+c\omega^3{}_4 \tag{3}$$

where $\omega^j{}_i$ are linear functions in the differentials du, dv and dw, $\omega^3{}_2$, $\omega^1{}_4$, $\omega^3{}_4$ are basis forms and a, b, c are invariants of the 1*st* differential neighbourhood of l. Exterior differentiation of (3) gives

$$\Delta a\Lambda\omega^3{}_2+\Delta b\Lambda\omega^1{}_4+\Delta c\Lambda\omega^3{}_4 = 0$$

where

$$\Delta \begin{bmatrix} a \\ b \\ c \end{bmatrix} = d \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} a^2 & -c-ab \\ c+ab & -b^2 \\ ac & -cb \end{bmatrix} \begin{bmatrix} \omega^3{}_1 \\ \\ \omega^2{}_4 \end{bmatrix} \tag{4}$$

Let us denote by δ, as usual, the differentiation such that $\delta u = \delta v = \delta w = 0$, and $\pi^j{}_i$ the form arising from $\omega^j{}_i$ by putting $du = dv = dw = 0$ in it. We have obviously the variation in the invariants

$$\delta \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a^2 & -c-ab \\ c+ab & -b^2 \\ ac & -cb \end{bmatrix} \begin{bmatrix} \pi^3{}_1 \\ \\ \pi^2{}_4 \end{bmatrix}$$

In general the matrix M; where

$$M = \begin{bmatrix} a^2 & -c-ab \\ c+ab & -b^2 \\ ac & -cb \end{bmatrix},$$

has a rank two. For some special classes of complexes, M has rank lower than two. We give complete classification of such complexes for which M has a rank zero and one at most.

The first class. For this class of complexes $R(M) = 0$, which gives $a = b = c = 0$ only, under this condition, equation (3) takes the form

$$\omega^1{}_2 = 0. \tag{5}$$

From the integrability of (5), it follows that, the line complex (5) exists within one constant in the first differential neighbourhood of the ray l.

THEOREM 1. *The line complex* (5) *can be stratified into one-parameter family of parabolic congruence.*

Proof. For the line complex $\omega^1{}_2 = 0$, the equation

$$\omega^3{}_2 = 0, \tag{6}$$

satisfies $D\omega^3{}_2 \equiv 0$, this means that, the differential equation (6) determines a holonomic line congruence (Kovancov 1963) belongs to the line complex (5), i.e., the line complex (5) represents a one-parameter family of holonomic line congruences.

To find the focal points of the line congruence (6), let $F = \bar{A}_2+t\bar{A}_4$ be any analytic point on the ray l. This point will be a focal point if $d\bar{F} \equiv 0 \pmod{\bar{F}}$, which gives the equalities

$$\omega^1{}_2+t\omega^1{}_4 = 0,\ \omega^3{}_2+t\omega^3{}_4 = 0,\ (\omega^1{}_2 = 0 = \omega^3{}_2).$$

This means that, the focal points are real and coincident with the vertex $\bar{A}_2(t = 0)$, i.e., the congruence (6) is a parabolic. In this case, the focal point $\bar{A}_2$ is fixed and the focal surface reduce to a point.

The line congruence (6) can be constructed as a two-dimensional subset of a boundle of lines with vertex $\bar{A}_2$ in the ideal plane.

LEMMA 1. *The line complex* (5) *is a special linear complex ; its axis is a fixed line on the ideal plane.*

Proof. From the differentials

$$d\bar{A}_2 = \omega^3{}_2\bar{A}_3,\ d(\bar{A}_2\bar{A}_3) = 0,\ \omega^1{}_2 = 0,$$

it follows, that, $\bar{A}_2$ describes the fixed line $(\bar{A}_2\bar{A}_3)$ which lie on the ideal plane $x^4 = 0$. Since all the rays of the line complex (5) intersect the fixed line. Hence the line complex (5) is a special linear complex with axis $x^1 = 0$, $x^4 = 0$.

For the line complex (5), the plane corresponding to any analytic point on the ray l in the normal correlation coincident with the coordinate plane $x^1 = 0$. i.e., the cone of rays of the line complex (5) will be a cylinder.

To obtain the equation of the line complex (5) in plücker coordinates, we consider the equation of the special linear complex in general form

$$a_{12}P^{34}+a_{13}P^{24}+a_{14}P^{23}+a_{23}P^{14}+a_{24}P^{13}+a_{34}P^{12} = 0 \tag{7}$$

where P^{ij}, $a_{ij}\,(i, j = 1, 2, 3, 4)$ are the plücker coordinates of the ray l and the axis of the line complex respectively and a_{ij} satisfies the plücker condition (Hlavaty 1953).

$$a_{34}a_{12}+a_{24}a_{13}+a_{24}a_{14} = 0.$$

Since the axis of the complex (5) generated by the ray $l = (\bar{A}_2\bar{A}_4)$ is a fixed line $(\bar{A}_2\bar{A}_3)$ and l intersect the coordinate planes $x^3 = 0$, $x^4 = 0$ in distinct two points, we can without loss of generality take the axis to be the line joining the two points $(1, \lambda, 0, 0)$ and $(1, 0, 1/\lambda, 0)$. Hence we have

$$a_{14} = a_{24} = a_{34} = 0,\ a_{12} = -\lambda,\ a_{13} = 1/\lambda,\ a_{23} = 1,\ P^{34} = 0, \tag{8}$$

introduce the values (8) in (7) we get the equation of the line complex (5) in plücker form

$$\lambda P^{14}+P^{24} = 0.$$

It is clear that, this equation determined within one arbitrary constant λ which is the parmeter of choosing the fixed line on the ideal plane. Finally we have to construct geometrically the line complex (5), as follows, the set of all lines in the space which intersect a fixed line on the ideal plane.

The second class. For this class $R(M) = 1$, this possible in the following cases (i) $c = 0,\ a = b \neq 0$, (ii) $a = 0,\ c = 0,\ b \neq 0$, (iii) $b = 0, c = 0, a \neq 0$.

For these cases, the differential equation of a line complex (3) in the 1*st* differential neighbourhood of the ray l takes the following forms respectively

$$\omega^{1}{}_{2} = a(\omega^{3}{}_{2}+\omega^{1}{}_{4}) \tag{9}$$

$$\omega^{1}{}_{2} = b\omega^{1}{}_{4} \tag{10}$$

$$\omega^{1}{}_{2} = a\omega^{3}{}_{2}. \tag{11}$$

LEMMA 2. *Every one of the line complexes* (9), (10) *and* (11) *exists within one function of one argument.*

Proof. Exterior differentiation of (9) and using Cartan's Lemma, we get

$$da - a^{2}(\omega^{3}{}_{1} - \omega^{2}{}_{4}) = \mu(\omega^{3}{}_{2}+\omega^{1}{}_{4}).$$

It follows that, the line complex (9) exists within one arbitrary function of one argument μ.

In order to complete the proof of the lemma, using a similar way for the complexes (10) and (11).

LEMMA 3. *The line complex* (9) *admits a fibration into* 2*-parameter families of line cones.*

Proof. For the line complex (9), the equations

$$\omega^{1}{}_{2} = 0,\ \omega^{1}{}_{4} = 0,\ \omega^{3}{}_{2} = 0,$$

determine a ruled surface and from the differentials

$$d\bar{A}_2 = 0,\ d(\bar{A}_2\bar{A}_4) = \omega^{3}{}_{4}(\bar{A}_2\bar{A}_3),$$

it follows that, the ray $l = (\bar{A}_2\bar{A}_4)$ generate a line cone (exists within one function of one argument) with vertex $\bar{A}_2$. Hence the complex (9) can be stratified into 2-parameter family of line cones with vertices on the ideal plane.

As revealed from the foregoing results, we give the theorem which gives the integral free-representation of the line complex (9).

THEOREM 2. *Take a cone of lines with vertex on the absolutum. The set of all line cones constructs the complex* (9).

Proof. We choose a frame of reference in the form of a coordinate tetrahedron $T(\bar{A}_1, \bar{A}_2, \bar{A}_3, \bar{A}_4)$, such that the vertices $\bar{A}_1$, $\bar{A}_2$ and $\bar{A}_3$ are placed on the invariant plane

$x^4 = 0$, $\bar{A}_3$ is the point of intersection of the two lines of the absolutum, $\bar{A}_4$ is a proper point and $\bar{A}_2$ is the vertex of the line cone generated by the ray $l = (\bar{A}_2\bar{A}_4)$.

The point $\bar{A}_2$ is a vertex of a cone iff (if and only if)

$$\omega^1{}_2 = 0, \ \omega^3{}_2 = 0,$$

the ray l generate a cone of lines iff the differential

$$dl = \omega^1{}_4(\bar{A}_2\bar{A}_1) + \omega^3{}_4(\bar{A}_2\bar{A}_3)$$

depends on one form, i.e., $\omega^1{}_4 = 0$, ($\omega^3{}_4 = 0$, not determine a ruled surface because of $\omega^3{}_4 = 0$, not integrable).

From $\omega^1{}_2 = 0$, $\omega^3{}_2 = 0$, $\omega^1{}_4 = 0$, we have

$$\omega^1{}_2 = 0 \quad \text{iff} \quad \omega^3{}_2 + \omega^1{}_4 = 0, \ D(\omega^3{}_2 + \omega^1{}_4) \equiv 0 \ (\text{mod } \omega^1{}_2)$$

i.e.,

$$\omega^1{}_2 = a(\omega^3{}_2 + \omega^1{}_4),$$

which is the differential equation of the line complex (9) in the 1*st* differential neighbourhood of the ray l.

THEOREM 3. *The plane corresponding to any analytic point on the ray l of the line complex* (9) *and* (11) *in the normal correlation is coincident with the planes corresponding to the points $\bar{A}_2$ and $\bar{A}_4$ of the complex* (9) *respectively.*

Proof. Let $\bar{P} = \bar{A}_2 + t\bar{A}_4$ be any analytic point on the ray $l = (\bar{A}_2\bar{A}_4)$ of a line complex, then

$$d\bar{P} = (\omega^1{}_2 + t\omega^1{}_4)\bar{A}_1 + (\omega^3{}_2 + t\omega^3{}_4)\bar{A}_3 + (dt - t^2\omega^2{}_4)\bar{A}_4 + t\omega^2{}_4\bar{P}$$

The point $\bar{P}$ is stable if $d\bar{P} \equiv 0$ (mod $\bar{P}$), i.e.,

$$\omega^1{}_2 + t\omega^1{}_4 = 0, \ \omega^3{}_2 + t\omega^3{}_4 = 0, \ dt - t^2\omega^2{}_4 = 0, \tag{12}$$

The differential equations (12) determine a line cone of a given complex generated by l.

The plane σ corresponding to $\bar{P}$ in the normal correlation has tangential coordinates of the form

$$\rho\sigma \equiv (\bar{A}_2\bar{A}_4\, d\bar{A}_2) \equiv \omega^1{}_2(\bar{A}_2\bar{A}_4\bar{A}_1) + \omega^3{}_2(\bar{A}_2\bar{A}_4\bar{A}_3), \tag{13}$$

where ρ is the coefficient of proportionality. For the line complex (9) and from (12), the equation (13) takes the form

$$\sigma \equiv at(\bar{A}_2\bar{A}_4\bar{A}_1) + (a+t)(\bar{A}_2\bar{A}_4\bar{A}_3). \tag{14}$$

The planes corresponding to the points $\bar{A}_2$ and $\bar{A}_4$ on the ray l of the line complex (9) in the normal correlation are

$$\sigma_{\bar{A}_2} \equiv (\bar{A}_2\bar{A}_4\bar{A}_3) \quad \text{and} \quad \sigma_{\bar{A}_4} \equiv a(\bar{A}_2\bar{A}_4\bar{A}_1) + (\bar{A}_2\bar{A}_4\bar{A}_3) \tag{14}$$

respectively. For the line complexes (10) and (11), the planes corresponding to $\bar{P}$ in the normal correlation are

$$\sigma_1 \equiv (\bar{A}_2\bar{A}_4\bar{A}_3) \quad \text{and} \quad \sigma_2 \equiv a(\bar{A}_2\bar{A}_4\bar{A}_1)+(\bar{A}_2\bar{A}_4\bar{A}_3), \tag{16}$$

respectively. From (15) and (16), the theorem is completely proved.

This theorem gives the geometrical meaning of the invariant a of the line complexes (9) and (11) in the 1*st* differential neighbourhood of the ray l where a is equal to the angle between the planes $\sigma_{\bar{A}_4}$ and $\sigma_{\bar{A}_2}$. The angle between the plane $\sigma_{\bar{A}_4}$ and the coordinate plane $(\bar{A}_2\bar{A}_4\bar{A}_1)$ is equal to the reciprocal of the invariant a mines one.

References

Finkov, S. P. (1948) : Cartan's methods of exterior forms and their applications in differential geometry, Moscow.

Hlavaty, V. (1953) : Differential line geometry Groningeen Nordhoff.

Kovancov, N. I. (1963) : Theory of complexes, Kiev.

Redei, L. (1968) : Foundation of Euclidean and non-Euclidean geometries, Hungary.

Strubecker, K. (1961) : Casi limiti di Geometri non-Euclidee Rendicondi del Seminario Matematica d'Universita e del Politecnico di Torino. **21**, 141

DEPARTMENT OF MATHEMATICS,
FACULTY OF SCIENCE,
ASSIUT UNIVERSITY, ASSIUT (EGYPT)

Bull. Cal. Math. Soc. **75**, 49–55 (1983)

SOME LOCAL PROPERTIES OF THE RAYLEIGH FUNCTIONS

E. C. Obi

(*Received* 22 *April* 1982)

Abstract. An application of the principle of reciprocal sequences and series to the Rayleigh functions leads to the identification of the coefficients of corresponding quotients of Bessel functions with local derivatives of the Rayleigh functions. In turn we get local recurrence formulas for these derivatives, as well as obtain a simplification of a formula for a sequence of Carlitz.

1. **Introduction.** In this article we study some local properties of the special functions, $\sigma_n(\nu)$, defined by $(\nu+n)\sigma_n(\nu) = \sum_{k=1}^{n-1} \sigma_k \sigma_{n-k}(\nu)$, $n > 1$, $\sigma_1(\nu) = \{4(\nu+1)\}^{-1}$. These rational functions, which are important and useful in various settings, arose originally as an aid in the study of the roots of Bessel functions. Since then they have appeared as coefficients in the series solutions of certain classes of Riccati differential equations. (See Kishore (1967), Obi (1978) and cfs. there for details and general properties.).

Given the polynominals, ϕ_n, in ν defined by

$$\phi_n(\nu) = \sum_{k=1}^{n-1} \Omega_k(\nu)\phi_n\, \phi_{n-k}(\nu),\ n > 1,\ \phi_1(\nu) = 1,$$

where

$$\Omega_k(\nu) = \prod_{s=1}^{n-1} (\nu+s)^{\epsilon(s,\,k,\,n)},\ \epsilon(s,\,k,\,n) \equiv \left[\frac{n}{s}\right] - \left[\frac{k}{s}\right] - \left[\frac{n-k}{s}\right],$$

$$[x] \equiv \text{greatest, integer function of } x,$$

then σ_n is of the form (Kishore 1964),

$$\sigma_n(\nu) = \phi_n(\nu)\left\{4^n \prod_{k=1}^{n} (\nu+k)^{[n/k]}\right\}^{-1}$$

On $\nu > -1$, it can be given an integral representation (Obi 1980) of the form,

$$\sigma_n(\nu) = \int_0^\infty e^{-\nu t}\, dg_n(t),$$

where g_n is a certain non-decreasing function on $[0, \infty)$, or of bounded variation on finite subintervals of $[0, \infty)$. An explicit formula for σ_n or ϕ_n is unknown. σ_n is called the Rayleigh function, and ϕ_n the Rayleigh polynomial, of order n. Let B_n denote the absolute value of the n-th Bernoulli number, G_n the n-th Genocchi integer, and a_n the numbers of Carlitz, viz.,

$$a_n = 2^{2n} n\,!\,(n-1)\,!\,\sigma_n(0), \quad G_n = 2(1-2^{2n})B_n.$$

Then the following useful local properties are known (Kishore, 1964; Carlitz 1963) :

$$\sigma_n(\tfrac{1}{2}) = -\frac{2^{2n-1}}{(2n)\,!}B_n, \quad \sigma_n(-\tfrac{1}{2}) = \frac{-2^{2n-2}}{(2n)\,!}G_n, \tag{$+$}$$

a_n is a positive integer, and for p prime, $a_{mp} \equiv a_m\,(p)$. (More congruence properties for a_n have been studied in Carlitz (1963), and the application of $(^+)$ to differential equations is in Kishore (1967). Otherwise no other basic local properties appear to be known. In what follows, we investigate some interesting local properties of $\sigma'_n(\nu)$ at $\nu = 0, 1$, derive another property of σ_n at the origin, and discuss a formula of Carlitz.

2. **Reciprocal sequences**. A real sequence, u_n, $n \geqslant 0$, with $u_0 = 1$, is said to have (or determine) a reciprocal sequence (*rs*), λ_n, $n \geqslant 1$, if $u_n = \sum_{k=1}^{n} \lambda_k u_{n-k}$. Note that if λ_n is the *rs* of u_n, and if $\lambda_n \geqslant 0\ \forall\ n \geqslant 1$, then $u_n \geqslant 0\ \forall\ n$, but that the converse fails. A real sequence $u_n > 0$, $n \geqslant 0$, with $u_0 = 1$, is called a generalized renewal sequence (*grs*) if its *rs*, λ_n, satisfies $\lambda_n \geqslant 0$. A power series, $U(z) = \sum_0^\infty u_n z^n$, with real coefficients, u_n, where $u_0 = 1$, is said to have or determine a reciprocal power series (*rps*), $\Lambda(z) = \sum_1^\infty \lambda_n z^n$, if $U = U\Lambda + 1$. The connection between *rps* and *rs* is that

$$u_n = \sum_{k=1}^{n} \lambda_k u_{n-k} \quad \text{iff } U(z) = U(z)\Lambda(z)+1,$$

where $U = \sum_0^\infty u_n z^n$, $\Lambda = \sum_1^\infty \lambda_n z^n$. The rs (rps) of $u_n(U)$ is unique for each sequence (power series); $\Lambda(z)$ is the difference between 1 and the formal reciprocal of $U(z)$. Alternatively, $\{U(z)\}^{-1} = 1-\Lambda(z)$ (cf. also Amer. Math. Monthly 65 (1958), p. 91.)

PROPOSITION (i) *Let $J_\nu(z)$ be the Bessel function of the first kind. If u_n, $n \geqslant 0$, is the sequence of coefficients of $J_0^2(\sqrt{z})$, then its rs is $2\sigma'_n(0)$, $n \geqslant 1$; hence the rps of $U(z) = J_0^2(\sqrt{z})$ is $\Lambda(z) = \sum_{n=1}^{\infty} 2\sigma'_n(0)z^n$;*

(ii) *the sequence $\lambda_n = \sigma'_n(1)$, $n \geqslant 1$, is the rs determined by the sequence of coeffcients of $\dfrac{2J_1(\sqrt{z})}{1-J_0^2(\sqrt{z})}$.*

To prove the preceding, we shall use the following basic lemma of Fubini (Randolph, 1968).

LEMMA. *If Σf_n is a pointwise convergent series of differentiable monotone-decreasing (alternatively monotone-increasing) functions on a closed interval* $[a, b]$, *then ,almost everywhere on* $[a, b]$, *term-by-term differentiation holds.*

Proof of the Proposition. Part (i) implies showing that

$$J_0^{-2}(z) = 1-\sum_{n=1}^{\infty} 2\sigma'_n(0)z^{2n} \quad (|z| < j_{0\cdot 1}). \tag{1}$$

Now consider the generating function [1;(2)],

$$J_{\nu+1}(x)/J_\nu(x) = \sum_{n=1}^{\infty} 2\sigma_n(\nu)x^{2n-1}, \tag{2}$$

for σ_n. For each n, and throughout $\nu > -1$, $\sigma_n(\nu) > 0$ and is a strictly monotone decreasing differentiable function of ν there, as may be checked readily from its recurrence equation. Choose and fix $x = \alpha$ arbitrarily in $|x| < 2$. (This choice is independent of ν.) Let $I = [0, 2]$ and $j_{\nu,1} =$ the first positive zero of $J_\nu(z)$. For all $\nu \epsilon I$, the roots of J_ν are real and the nonzero root nearest the origin is more than 2 units off, so that $j_{\nu,1} > 2$, and the constant α therefore belongs to all the intervals $D_\nu \equiv (|x| < j_{\nu,1})$, $\nu \epsilon I$, of convergence of (2) independent of the changes in ν over I. Then not only does the particular equation,

$$J_{\nu+1}(\alpha)/J_\nu(\alpha) = \sum_{n=1}^{\infty} 2\alpha^{2n-1}\sigma_n(\nu), \tag{2a}$$

remain valid as ν varies in I, but also, in view of the monotonic properties of σ_n noted before, the function of ν defined on I by $f_{n,\alpha}(\nu) = 2\alpha^{2n-1}\sigma_n(\nu)$ will be *either* definitely monotone decreasing in ν, for each n, *or* definitely monotone increasing in ν, for each n, according as $\alpha \geqslant 0$ or $\alpha < 0$. In either monotonic condition, the lemma implies that the series (2a) is term-by-term differentiable with respect to ν on I, to yield $(d/d\nu)\{J_{\nu+1}(\alpha)/J_\nu(\alpha)\}$ on I. Then, evaluating this derivative at $\nu = 0$, we find that if $\alpha \neq 0$,

$$\sum_{n=1}^{\infty} 2\sigma'_n(0)\alpha^{2n-1} = J_0^{-2}(\alpha)\left\{ J_0(\alpha)\left[\frac{d}{dt} J_t(\alpha)\right]_{t=1} -\tfrac{1}{2}\pi J_1 Y_0(\alpha)\right\}$$

$$= \alpha^{-1}+\tfrac{1}{2}\pi J_0^{-2}(\alpha)[J_0 Y_\nu(\alpha)-J_1 Y_0(\alpha)],$$

where $Y_\nu(z)$ is the Weber function. However, from the Wronskian identity,

$$J_\nu Y_{\nu+1}(z)-J_{\nu+1} Y_\nu(z) = -2/(\pi z),$$

the above reduces to

$$\sum_{n=1}^{\infty} 2\sigma'_n(0)\alpha^{2n-1} = \alpha^{-1}(1-J_0^{-2}(\alpha));$$

whence,

$$J_0^{-2}(\alpha) = 1 - \sum_{n=1}^{\infty} 2\sigma'_n(0)\alpha^{2n}. \tag{3}$$

Since $x = \alpha$ is arbitrarily fixed in $|x| < 2$, (3) will then give the desired Taylor expansion throughout $|x| < j_{0\,1}$. This proves (i).

To obtain (ii), put the identity $J_{\nu+1}(x) = (2\nu/x)J_\nu(x) - J_{\nu-1}(x)$ into (2). Then,

$$(2\nu/x) - J_{\nu-1}(x)/J_\nu(x) = \sum_{n=1}^{\infty} 2\sigma_n(\nu)x^{2n-1}, \tag{4}$$

from which therefore, by exactly the same argument that followed (2), we obtain on differentiating at $\nu = 1$ on I,

$$\alpha^{-1}\{2 - J_1^{-2}(\alpha)[1 - J_0^2(\alpha)]\} = \sum_{n=1}^{\infty} 2\sigma'_n(1)\alpha^{2n-1}.$$

This gives the following expansion whose right side should be compared to that of (1):

$$\frac{1 - J_0^2(\alpha)}{J_1^2(\alpha)} = 2 - \sum_{n=1}^{\infty} 2\sigma'_n(1)\alpha^{2n}, \tag{4a}$$

which will be valid throughout $|\alpha| < j_{1,\,1}$. Now (ii) follows on dividing (4a) by 2.

Note. (a) On observing that for all $n \geqslant 1$, $\sigma_n(\nu)$ is analytic throughout $\operatorname{Re}\nu > -1$, it is natural to seek a subregion of the ν- and z-planes where the order of the double summation,

$$\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} b_{nm}(\nu - \nu_0)mz^{2n-1}, \quad b_{nm} \equiv 2m\,!^{-1}\sigma_n{}^{(m)}(\nu_0), \tag{5}$$

is interchangeable in neighbourhoods of $(\nu_0, 0)$, because in such neighbourhoods (5) would give the double series for $J_{\nu+1}(z)/J_\nu(z)$. Using our recurrence relation, an induction on n shows that $|\sigma_n(\nu)| \leqslant (1/4)^n$ on the 1/2 plane, $D \equiv (\operatorname{Re}\nu \geqslant 0), \forall\, n \geqslant 1$. Fixing α in $V \equiv (|z| < 2)$, the preceding implies that our former function $f_{n,\alpha}(\nu)$ will be here uniformly bounded on D, by $2|\alpha|^{2n-1}4^{-n}$. It follows by (Weierstrass M-test) that $\sum_{n=1}^{\infty} f_{n,\alpha}(\nu)$ is uniformly and absolutely convergent on D and hence in every closed, proper, and concentric subdisc of $(|\nu - \nu_0| < \operatorname{Re}\nu_0) \equiv W \subset D^0$.

Now impose Weierstrass double-series theorem to $F \equiv \sum_{n=1}^{\infty} f_{n,\alpha}(\nu)$ in W, and we have

$$F = \sum_{m=0}^{\infty} \left\{ \sum_{n=1}^{\infty} m\,!^{-1} f_{n,\alpha}^{(m)}(\nu_0) \right\} (\nu - \nu_0)^m$$

$$= \sum_{m=0}^{\infty} \left\{ \sum_{n=1}^{\infty} m\,!^{-1} 2\alpha^{2n-1} \sigma_n^{(m)}(\nu_0) \right\} (\nu-\nu_0)^m$$

$$= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} b_{nm}(\nu-\nu_0)^m \alpha^{2n-1}.$$

But also in W, we have the series,

$$f_{n,\alpha}(\nu) = \sum_{m=0}^{\infty} m\,!^{-1} f_{n.\alpha}^{(m)}(\nu_0)(\nu-\nu_0)^m = \sum_{m=0}^{\infty} b_{nm}\alpha^{2n-1}(\nu-\nu_0)^m,$$

so that from definition,

$$F = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} b_{nm}(\nu-\nu_0)^m \alpha^{2n-1}.$$

Thus the order of the given summation is interchangeable in $W \times V$, and the double series represents F:

$$zJ_{\nu+1}(z)/J_\nu(z) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} b_{nm}(\nu-\nu_0)^m z^{2n},$$

valid at least in $W \times V$.

(b) It may be pointed out, too, that (2} and its reciprocal from (4) (put $\nu = \nu+1$ in (4)) can be extended by using Lommel polynomials, $R_{m,\nu}$, to yield coefficients in terms of σ_n for the more general quotients, $(J_{\nu+n}/J_{\nu+k})(z)$. For $J_{\nu+m}$ and $R_{m,\nu}$ are related by the identities (Watson 1944),

$$J_{\nu+m}(z) = (J_\nu R_{m,\nu})(z) - (J_{\nu-1} R_{m-1,\nu+1})(z), \tag{5a}$$

$$J_\nu(z) = -(J_{\nu+m} R_{m-2,\nu+1})(z) + (J_{\nu+m-1} R_{m-1,\nu+1})(z). \tag{5b}$$

Example. $(J_{\nu-2}/J_{\nu-4})(z) = \dfrac{J_{(\nu-4)+2}}{J_{\nu-4}}(z) = \sum_{n=1}^{\infty} 4(\nu-3)\sigma_{n+1}(\nu-4)z^{2n}.$

3. Local $\sigma'_n(\nu)$ **Formulas.** Formulae for the recursive calculations of the derivative at the origin and $\nu = 1$, are almost immediate from the proposition. Let u_n, $n \geqslant 0$, be the coefficients of $J_0^2(\sqrt{z})$. The proposition says that

$$\sum_{k=1}^{n} 2u_{n-k}\sigma'(0) = u_n.$$

But from the following identity (Watson 1944),

$$J_\nu^2(z) = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(2\nu+2n+1)z^{2\nu+2n}}{n\,!\,2^{2\nu+2}\,\Gamma(2\nu+n+1)\Gamma^2(\nu+n+1)},$$

we see that $u_n = (-1)^n \frac{(2n)\,!}{4^n n\,!^3}$. Therefore,

$$\sum_{k=1}^{n} \frac{(-1)^{n-k}(2n-2k)\,!}{2^{2n-2k-1}(n-k)\,!^3}\;\sigma'_k(0) = (-1)^n \frac{(2n)\,!}{4^n n\,!^3} (n \geqslant 1). \tag{7}$$

On the other hand, if in (1) we replace z by $\sqrt{z}$ and multiply both sides by $J_0^2(\sqrt{z})$ using (6), then on identifying the coefficient of z^n we obtain,

$$\sigma'_n(0) + \sum_{k=1}^{n-1} \frac{(-1)^k(2k)\,!}{4^k k\,!^3} \sigma'_{n-k}(0) = (-1)^n \frac{(2n)\,!}{2^{2n+1}\,n\,!^3} \quad (n > 1), \tag{7a}$$

with $\sigma'_1(0) = 1/4$. Similarly using (4a) and (6), we obtain a recursion formula for $\sigma'_n(1)$:

$$\sum_{k=1}^{n-1} \beta_k \sigma'_k(1) = -\tfrac{1}{2}\beta_0((n-1)\,!\,(n+1)-2),$$

where

$$\beta_k = \frac{(-1)^{n-k-1}4^k(2n-2k)\,!}{(n-k-1)\,!\,(n-k+1)\,!\,(n-k)\,!^2},\ 0 \leqslant k \leqslant n-1,\ n > 1.$$

4. **Remarks**. Square both sides of (2), then multiply by $J_\nu^2(z)$ and substitute (6). By comparing the coefficients of z^{2n}, it can be shown (Kishore 1963), using the recursive definition of σ_n in Section 1, that

$$\sum_{k=1}^{n} (-1)^{k-1}4^k \binom{2\nu+2n-2k}{n-k} (k-1)\,!^2 \binom{\nu+n-1)}{|k-1}^2 (\nu+k)\sigma_k(\nu)$$

$$= \frac{2\nu+1}{2\nu+n} \binom{2\nu+2n-2}{n-1}.$$

Set $\gamma_k \equiv (-1)^{k-1}4^k \binom{2n-2k}{n-k} (k-1)\,!^2 \binom{n-1}{k-1} k$, for fixed n. Then,

$$\sum_{k=1}^{n} \gamma_k \sigma_k(0) = \frac{1}{n} \binom{2n-2}{n-1} \equiv e_n. \tag{8}$$

The number e_n, the reader may recognize, is the n-th Catalan number, an important well-known integer in combinatorics. (In ring theory, e_n counts the ideals in the ring of upper triangular matrices; it is the leading coefficient of our polynomial, ϕ_n, of Section 1; cf. (Obi 1978) and its refs. In this connection, it is of interest that (8) shows how e_n can be identified with certain sums and differences of the first n Rayleigh functions at the origin.) (8) also connects the integers of Carlitz, a_k, with e_n, viz.

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{2n-2k}{n-k} \binom{n-1}{k-1}^2 a_k = e_n.$$

As a second remark, consider the following sequence, defined by Carlitz (1963) implicitly and recursively in terms of a_n :

$$c_1 = 2, \quad c_n = \frac{2}{n} a_n + \sum_{k=1}^{n-1} 2 \binom{n-1}{k} \binom{n-1}{k-1} a_k \, c_{n-k}, \quad (n > 1).$$

Since $a_n = 2^{2n} n\,!\,(n-1)\,!\,\sigma_n(0)$, does there exist a similarly simple and explicit formula for c_n in terms of $\sigma_n(0)$ or (perhaps) $\sigma'_n(0)$? Set $\lambda_0 = 1$, and $\lambda_n = c_n/(4^n n(n-1)\,!^2)$, $n \geqslant 1$. Carlitz (1963) showed that λ_n, $n \geqslant 0$, is identifiable with the coefficients of $\{J_0^2(\sqrt{z})\}^{-1}$. Then $-\lambda_n$, $n \geqslant 1$, is the *rs* of the coefficients of $J_0^2(\sqrt{z})$. Applying our Proposition to this result, we get $\lambda_n = -2\sigma'_n(0)$, $n \geqslant 1$, and hence the following beautiful analogue between c_n and the definition of a_n :

$$a_n = 2^{2n} n\,!\,(n-1)\,!\,\sigma_n(0),$$

$$c_n = -(2^{2n+1}\, n\,!\,(n-1)\,!)\sigma'_n(0).$$

This answers the question in the affirmative.

References

Carlitz, L. (1963) : A Sequence of integers related to Bessel functions; *Proc. A.M.S.* **14**, 1, 1.
Kishore, N. (1967) : A class of formulas for the Rayleigh function; *Duke M. Jnl.* **34**, 3, 573.
——— (1964) : A representation of Bernoulli numbers; *Pacific Jnl. Math.*, **14**, 4, 1297.
——— (1963) : The Rayleigh function; *Proc. Amer. Math. Soc.*, **14**, 4, 527.
Obi, E. C. (1978) : A Note on the Rayleigh polynomial; *SIAM Jnl. Math. Anal.* **9**, 5, 825.
——— (1980) : Complete Monotonicity of the Rayleigh function; *Jnl. Math. Anal. & Appl.* **77**, 2, 465.
Randolph, J. F. (1968) : *Basic real and abstract analysis*; Academic Press Inc., N.Y. & London, 413.
Watson, G. N. (1944) : *A Treaties....Theory of Bessel function*; MacMillan, N.Y., 502.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIGERIA
NSUKKA, NIGERIA.

Bull. Cal. Math. Soc. **75**, 57–58 (1983).

A NEW CHARACTERIZATION OF PERFECT AND SEMI-PERFECT RINGS

JUTTA HAUSEN AND JOHNNY A. JOHNSON[1]

(*Received* 28 *January* 1980)

Abstract. A new characterization is given of left perfect and semi-perfect rings with identity in terms of their module classes.

Left perfect and left semi-perfect rings were introduced by Hyman Bass in (Bass 1960). Two module-theoretical concepts, which we shall call "supplemented" and "amply supplemented", have been considered by a number of authors using different terminology ("komplementiert" in (Hauger 1976, Kasch 1977, 1966, Wisbauer 1978); "supplemented" in (Golan 1971) and "perfect" in (Miyashita 1966)).

The purpose of this note is to give characterizations of the left perfect and semi-perfect rings in terms of their supplemented and amply supplemented modules.

Throughout, R is a ring with identity and all modules are unital left R-modules. A module M is said to be supplemented if, for every submodule N of M, there exists a submodule S of M which is minimal with respect to the property that $M = N+S$; if, given any pair of submodules N and U of M with $M = N+U$, there exists a submodule S of U which is minimal with respect to $M = N+S$, then M is called amply supplemented.

We shall prove the following two theorems.

THEOREM 1. *The following statements are equivalent.*

(i) *R is left perfect.*

(ii) *Every left R-module is supplemented.*

(iii) *Every left R-module is amply supplemented.*

THEOREM 2. *The following statements are equivalent.*

(i) *R is left semi-perfect* .

(ii) *Every finitely generated left R-module is supplemented.*

(iii) *Every finitely generated left R-module is amply supplemented.*

These characterizations show a remarkable analogy to Bass's original definitions: R is left perfect if every left R-module has a projective cover, and R is left semi-perfect if every finitely generated left R-module has a projective cover (Bass 1960, pp. 467, 471).

---

[1] The second author was partially supported by a University of Houston Central Campus Research Grant.

Theorems 1 and 2 suggest that supplemented and amply supplemented modules deserve attention similar to that given modules with projective covers.

Since a ring is left semi-perfect if and only if it is right semi-perfect (Bass 1960, p. 471, Theorem 2.1), Theorem 2 can be expanded. Numerous other characterizations of perfect and semi-perfect rings are well known (Anderson and Fuller 1974; Bass 1960; Faith 1976).

We remark that supplemented modules in general need not be amply supplemented (Hausen and Johnson).

Proof of Theorem 1. In order to derive (iii) from (i), let M be a left R-module and let P be a projective module such that M is an epimorphic image of P. By (i), every epimorphic image of P has a projective cover. It follows from (Miyashita 1966; p. 96, Theorem 3,3 and p. 90, Theorem 1.12) that M is amply supplemented. Thus, (i) implies (iii). Trivially, (iii) implies (ii). Assume (ii), let M be a left R-module, and let P be a projective R-module such that M is an epimorphic image of P. Combining (ii) with (Kash and Mares 1966; p. 526, SATZ), it follows that every epimorphic image of P has a projective cover. Hence M has a projective cover, and the proof is completed.

Proof of Theorem 2. In the proof of Theorem 1, choose M and P to be finitely generated.

References

Anderson, F. W. and Fuller, K. R. (1974): "Rings and Categories of Modules". *Springer-Verlag*, New York.

Bass, H. (1960): Finitistic dimension and a homological generalization of semi-primary rings. *Trans. Amer. Math. Soc.* 95, 466.

Faith, C. (1976): "Algebra II Ring Theory", *Springer-Verlag*, Berlin.

Golan, J. S. (1971): Quasiperfect modules, *Quart. J. Math.* Oxford (2), 22, 173.

Hauger, G. (1976): Aufsteigende Kettenbedingung für zyklische Moduln und perfekte Endomorphismenringe, *Acta Math. Acad. Sci.* Hungar, 28, 275.

Hausen, J. and Johnson, J. A. A characterization of two classes of Dedekind domains by properties of their modules, (to appear).

Kasch, F. (1977): "Moduln und Ringe", B. G. Teubner, Stuttgart.

Kasch, F. and Mares, E. A. (1966): Eine Kennzeichnung semi-perfekter Moduln, *Nagoya Math. J.* 28, 525.

Miyashita, Y. (1966): Quasi-projective modules, perfect modules, and a theorem for modular lattices, *J. Fac. Sci.* Hokkaido Univ. 19 86.

Wisbauer, R. (1978): σ -semiperfekte und σ-perfekte Moduln, *Math. Z.* 162, 131.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HOUSTON, CENTRAL CAMPUS
HOUSTON, TEXAS 77004

Bull. Cal. Math., Soc., **75**, 59-62 (1983)

RELATION BETWEEN TOPOLOGY AND PROPERTIES OF CURVATURE OPERATOR OF A RIEMANNIAN MANIFOLD

GRIGORIOS TSAGAS

(*Received* 1 *June* 1979)

1. Introduction We consider a compact orientable Riemannian manifold of dimension n. Let $T_p(M)$ be the tangent space of M at the point P. The curvature tensor field R_1 at the point P and the two vectors $X, Y \in T_p(M)$ induce an endomorphism on $T_p(M)$ as follows :

$$R_1(X, Y) : T_p(M) \to T_p(M), \quad R_1(X, Y) : Z \to R_1(X, Y)Z.$$

If X, Y, Z, W are four vectors of $T_p(M)$, the Riemannian curvature tensor field R at P can be considered as a quadrilinear mapping, which is given by :

$$R : T_p(M) \times T_p(M) \times T_p(M) \times T_p(M) \to \boldsymbol{R},$$

$$R : (X, Y, Z, W) \to <R_1(X, Y)Z, W>.$$

If X, Y are two linearly independent vectors of $T_p(M)$, then the sectional curvature of the plane λ which is spanned by X, Y, is given by :

$$\sigma(X, Y) = \sigma(\lambda) = -< R_1(X, Y)X, Y > / \|X\|^2 \|Y\|^2 - < X, Y >^2.$$

2. We consider the dual space $T^*_p(M)$ of $T_p(M)$. Let $\overset{2}{\Lambda} T^*_p(M)$ be the vector space of exterior 2-forms at the point P. On the vector space $\overset{2}{\Lambda} T^*_p(M)$ we define a linear endomorphism q as follows :

$$q : \overset{2}{\Lambda} T^*_p(M) \to \overset{2}{\Lambda} T^*_p(M), \; q : \omega \to q(\omega)$$

$$q : \omega \to q(\omega), \; q(\omega)(X, Y) = -\text{trace}_z \, \omega(R_1(X, Y)Z, Z),$$

where $X, Y, Z \in T_p(M)$.

The endomorphism q is called curvature operator at the point P of the Riemannian manifold M. By the duality between $\overset{2}{\Lambda} T^*_p(M)$ and $\overset{2}{\Lambda} T_p(M)$ defined by the Riemannian metric we also obtain a similar endomorphism q on the space $\overset{2}{\Lambda} T_p(M)$.

The aim of the present paper is to study the influence of some properties of the curvature operator q on the topology and the geometry of the Riemannian manifold.

In the paragraph two we give some basic definitions and known results, in paragraph 3 the vector space $K^p(M, \boldsymbol{R})$, $p = 1, 2, \ldots, n-1$, of the Killing p-forms on the manifold M

is study, when the curvature operator q has all the eigenvalues negative. In the paragraph 4 we compare the dimensions of the vector spaces $K^p(M, R$ and $H^p(M, R)$, where $H^p(M, R)$ is the vector space of harmonic p-forms, in relation with some properties of the curvature operator q.

3. It is known that the curvature operator q is symmetric with respect to the inner product on $\overset{2}{\Lambda} T^*_p(M)$ induced by the Riemannian metric on M. From this we conclude that all the eigenvalues of q are real. The inequalities $K \leqslant q(M) \leqslant L$ means that all the eigenvalues of q at each point of M are contained in the interval $[K, \Lambda]$. Let $\{X_i\}$ be an orthonormal basis of the tangent space $T_p(M)$. This basis has a dual basis $\{X^*_i\}$ of $T^*_p(M)$. It can be easily proved :

$$< q(X^*_1 \wedge X^*_2), X^*_1 \wedge X^*_2 > = 26(X_1, X_2). \tag{3.1}$$

The inequality $K \leqslant q(M) \leqslant \Lambda$ yields $K/2 \leqslant \sigma \leqslant \Lambda/2$. The inverse is not true. We give a counter example. Let $P^n(C)$ be the complex projective space with Fubini-study metric whose holomorphic sectional curvature is equal 1 and therefore its sectional curvature satisfies the inequalities $1/4 \leqslant \sigma \leqslant 1$. Let ω be the Kaehler exterior 2-form on $P^n(C)$, then $q(\omega) = 0$, (Berger, 1961).

An exterior p-form ω is called Killing p-form, if it satisfies the relation :

$$\nabla_x \omega(Y, X_2, \ldots, X_p) + \nabla_y \omega(X, X_2, \ldots, X_p) = 0, \tag{3.2}$$

where $X, Y, X_i \in T(M)$, $i = 2, \ldots, p$. Let (U, φ) be a chart on the manifold M. Let $(x^1, \ldots, x^n)$ be a local coordinate system in U. Let $(\omega_{i_1 \ldots i_p})$ be the components of a p-form with respect to $(x^1, \ldots, x^n)$. On U we consider the following quadratic form (Yano and Bochner, 1953).

$$F(\omega/U) = R_{ij}\omega^{ii_2 \ldots i_p}\, \omega^{j}_{\;i_2 \ldots i_p} + \frac{p-1}{2} R_{ijkl}\omega^{iji_3 \ldots i_p}\, \omega^{kl}_{\;\;i_3 \ldots i_p}. \tag{3.3}$$

If ω is a Killing p-form then, its components $(\omega_{i_1 \ldots i_p})$ satisfy the relation :

$$\nabla_j \omega_{i\, i_2 \ldots i_p} + \nabla_i \omega_{j\, i_2 \ldots i_p} = 0. \tag{3.4}$$

If ω is any Killing p-form, then we have (Yano and Bochner, 1953).

$$\Delta(|\omega|^2) = (|\nabla\omega|)^2 - F(\omega). \tag{3.5}$$

If we assume that the curvature operator q satisfies the inequality $-K \leqslant q(M) \leqslant -L < 0$, then $F(\omega)$ which is given by (3.3) and by means of (3.1) and after some calculations we have (Tsagas 1971).

$$F(\omega) \leqslant \frac{1}{2}\,[(p-1)K - (n-1)L]\,|\omega|^2. \tag{3.6}$$

If we set :

$$(p-1)K-(n-1)L < 0,$$

then this implies :

$$\frac{p-1}{n-1} < \frac{L}{K}. \tag{3.7}$$

If $L = K-\epsilon$, $\epsilon > 0$, then the inequality (3.7) becomes :

$$p < n - \frac{\epsilon}{K}(n-1). \tag{3.8}$$

If $\epsilon < \frac{K}{n-1}$, then we can set :

$$L = K - \frac{K}{n-1} + \epsilon_1 K, \ \epsilon_1 < \frac{1}{n-1}. \tag{3.9}$$

The inequality (3.7) by (3.9) becomes :

$$p < n-1+\epsilon_1(n-1). \tag{3.10}$$

The inequality (3.10) implies :

$$F(\omega) < 0 \text{ for every } p = 1, 2, \ldots, n-1. \tag{3.11}$$

From (3.5) and (3.11) we conclude the following theorem.

THEOREM 3.1. *Let M be a compact orientable Riemannian manifold. If the eigen-values of q are negative and belong to the open interval of length* $1/(1-n)$, *then* $K^p(M, \mathbf{R}) = 0$ *for every* $p = 1, 2, \ldots, n-1$.

4. Now we can state the following conjecture.

CONJECTURE. 4.1. *Let M be a compact orientable Riemannian manifold. If the eigen-values of q are positive, then* $\dim H^p(M, \mathbf{R}) \leqslant \dim K^p(M, \mathbf{R})$, $p = 1, 2, \ldots, n-1$, $n = \dim M$. *If the eigen-values of q are negative, then* $\dim H^p(M, \mathbf{R}) \geqslant \dim K^p(M, \mathbf{R})$, $p = 1, 2, \ldots, n-1$.

The following theorems have been proved.

THEOREM 4.2. (Yano and Bochner, 1953), (Berger, 1971). *Let M be a compact orientable Riemannian manifold whose curvature operator is positive, then* $H^p(M, \mathbf{R}) = 0$, $p = 1, 2$.

THEOREM 4.3. (Yano and Bochner, 1953), (Tsagas, 1969). *We consider a compact orientable Riemannian manifold whose curvature operator is negative, then* $K^p(M, \mathbf{R}) = 0$, $p = 1, 2$.

From the theorems (4.2) and (4.3) we conclude that the conjecture is true for $p = 1, 2$.

The first part of the conjecture has been proved under the following assumption (Yano and Bochner 1953).

THEOREM 4.4. *Let M be a compact orientable Riemannian manifold. If the curvature operator q has positive eigen-values in the open interval of length $1/2$, then $H^p(M, \mathbf{R}) = 0$, $p = 1, 2, \ldots, n-1$. Therefore dim $H^p(M, \mathbf{R}) \leqslant$ dim $K^p(M, \mathbf{R})$, $p = 1, 2, \ldots, n-1$.*

THEOREM (3.2) proves the second part of the conjecture under the assumption of this theorem.

References

Berger, M. (1967) : Sur les variétés à opérateur de courbure positif. *C.R. Acad. Sci.* Paris, **253**, 2832.

Chern, S. S. (1966) : The geometry of *G*-structures. *Bull. Amer. Math. Soc.*, **72**, 167.

Kobayashi, S. and Nomizu, K. (1969) : Foundations of Differential Geometry, **2**, New York, Interscience.

Tsagas, Gr. (1971) : Some properties of the curvature operator of a Riemannian manifold. *Math. Z* **119**, 283, *Springer-Verlag*.

Yano, K. and Bochner, S. (1953) : Curvature and Betti numbers. *Ann. of Math. Studies* **32**. Princeton. Univ. Press.

DEPARTMENT OF MATHEMATICS
SCHOOL OF TECHNOLOGY
UNIVERSITY OF THESSALONIKI
THESSALONIKI, GREECE

Bull. Cal. Math. Soc. 75, 63-68 (1983)

CONVERGING FACTORS FOR CERTAIN CONTINUED FRACTIONS $K(a_n/1)$, where $|a_n| < 1$

JOHN GILL

(*Received* 7 *September* 1981)

Abstract. The first results concerning the use of converging factors to enhance the convergence of non-limit periodic continued fractions are presented.

The approximants, A_n/B_n, of the continued fraction

$$\frac{a_1}{1}+\frac{a_2}{1}+\cdots+\frac{a_k}{1}+\cdots, \tag{1}$$

where each a_k is a non-zero complex number, can be described in the following manner. Let $t_n(z) = a_n/(1+z)$, $n \geqslant 1$ and $T_1(z) = t_1(z)$, $T_n(z) = T_{n-1}(t_n(z))$, $n \geqslant 2$. Then $A_n/B_n = T_n(0)$, $n \geqslant 1$.

Complex numbers $\mu_1, \mu_2, \ldots$ having the property that $\lim\limits_{n\to\infty} T_n(\mu_n) = \lim\limits_{n\to\infty} T_n(0)$, provided the latter limit exists, are called *converging factors* of (1).

Recently, several papers have been written concerning converging factors of limit-periodic continued fractions ($a_n \to a$). Gill (1975), Thron and Waadeland (1980) and Waadeland (1966) are examples, illustrating the avenues of approach to this sort of research. Very little has been revealed, however, about converging factors of continued fractions (1) that are *not* limit-periodic. Two results pertaining to these more general continued fractions are presented in this paper : 1. Using an elementary geometrical argument, a large class of converging factors is discovered for continued fractions satisfying the conditions of a classical theorem of Pringsheim (Perron, 1957). 2. For certain continued fractions of this type, converging factors actually improving convergence are investigated. A numerical estimate of the magnitude of improvement is obtained.

1. Converging Factors of (1).

Let $I_n = \{z : |z+1| \leqslant \sqrt{|a_n|}\}$, $n \geqslant 1$ and set $T = \lim\limits_{n\to\infty} T_n(0)$.

THEOREM 1. *If*

(i) $\sqrt{|a_n|}+\sqrt{|a_{n-1}|} \leqslant 1$, $n \geqslant 1$

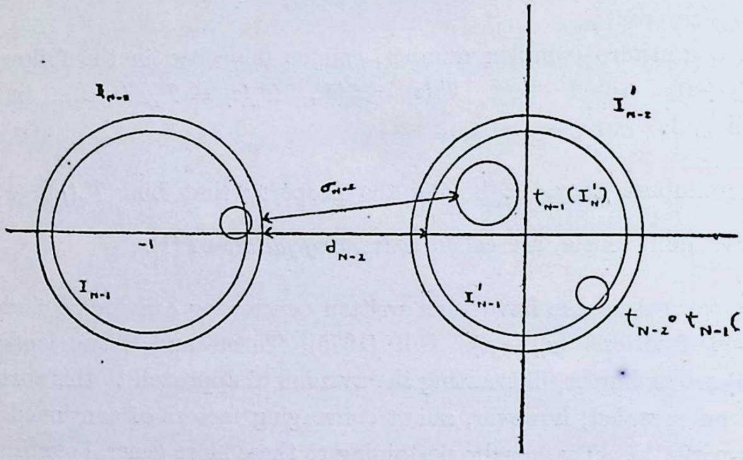
and

(ii) $\prod\limits_{1}^{n} \dfrac{\sqrt{|a_{j-1}|}}{1-\sqrt{|a_j|}} \to 0$ *are satisfied,* *then,* 1. *converges,* 2. *Complex numbers* $\{\mu_n : \mu_n \notin I_n, n \geqslant 1\}$ *are converging factors of* (1), (1) *and*

$$3. \quad |T_n(\mu_n)-T| < 2\sqrt{|a_n|} \prod_2^n \frac{|a_{j-1}|}{(1-\sqrt{|a_j|})^2}, \ n \geqslant 2.$$

Proof. I_n is the isometric circle of $t_n(z)$ (Ford 1951). Its radius is $\sqrt{|a_n|}$. Geometrically, the action of t_n upon z consists of (a) an inversion in I_n, (b) a reflection in the line $\{z : \mathrm{Re}(z) = -1/2\}$, and, usually, (c) a rotation about $z = 0$.

If $\mu_n \notin I_n$, then $t_n(\mu_n)$ lies within the inverse circle $I'_n = \{z : |z| \leqslant \sqrt{|a_n|}\}$, which is the image of I_n by t_n. The condition (i) guarantees that $t_n(\mu_n) \notin I_{n-1}$ and, more generally, that $I'_n \bigcap I_{n-1} = \phi$.



Now, both $t_n(0)$ and $t_n(\mu_n)$ lie in I'_n. Hence, $t_{n-1} \circ t_n(0)$ and $t_{n-1} \circ t_n(\mu_n)$ lie in I'_{n-1}. The image of the circle I'_n under t_{n-1} is a circle $t_{n-1}(I'_n) \subset I'_{n-1}$.

Similarly, $t_{n-2} \circ t_{n-1}(I'_n)$ is a circle contained in I'_{n-2}. The figure illustrates the case in which $I_{n-1} \subset I_{n-2}$; the other case can be treated analogously.

From (a), we have

$$\mathrm{Rad}[t_{n-2} \circ t_{n-1}(I'_n)]$$

$$= \mathrm{Rad}[t_{n-1}(I'_n)]. \ \frac{(\sqrt{|a_{n-2}|})^2}{(\sqrt{|a_{n-2}|}+\sigma_{n-2})(\sqrt{|a_{n-2}|}+\sigma_{n-2}+2\,\mathrm{Rad}[t_{n-1}(I'_n)])}$$

$$< \mathrm{Rad}[t_{n-1}(I'_n)]. \ \frac{|a_{n-2}|}{(\sqrt{|a_{n-2}|}+\sigma_{n-2})^2}$$

$$\leqslant \mathrm{Rad}[t_{n-1}(I'_n)]. \frac{|a_{n-2}|}{(\sqrt{|a_{n-2}|}+d_{n-2})^2}$$

$$= \mathrm{Rad}[t_{n-1}(I'_n)]. \frac{|a_{n-2}|}{(1-\sqrt{|a_{n-1}|})^2} .$$

Repeating this process gives

$$\mathrm{Rad}[T_{n-1}(I'_n)] < \mathrm{Rad}[t_{n-1}(I'_n)] \prod_{j=2}^{n-1} \frac{|a_{j-1}|}{(1-\sqrt{|a_j|})^2} .$$

Since

$$\mathrm{Rad}[t_{n-1}(I'_n)] < \sqrt{|a_n|} . \frac{|a_{n-1}|}{(1-\sqrt{|a_n|})^2} ,$$

we have

$$\mathrm{Rad}[T_{n-1}(I'_n)] < \sqrt{|a_n|} . \prod_{j=2}^{n} \frac{|a_{j-1}|}{(1-\sqrt{|a_j|})^2} . \tag{2}$$

$$T_n(\mu_n) \subset T_{n-1}(I'_n),\ T_n(0) \subseteq T_{n-1}(I'_n),\ T_n(I'_{n+1}) \subset T_{n-1}(I'_n),$$

and

$$\mathrm{Rad}[T_{n-1}(I'_n)] \to 0 \text{ as } n \to \infty$$

imply

$$\lim_{n\to\infty} T_n(\mu_n) = \lim_{n\to\infty} T_n(0) = T \in T_{m-1}(I'_m),\ m \geqslant 1.$$

Since $T_n(\mu_n)$ and T lie within $T_{n-1}(I'_n)$, assertion 3 of theorem 1 follows from (2).

2. Improvement of Convergence. We shall consider a_n's in (1) of the form $a_n = \alpha_n(\alpha_{n+1})$, where $|\alpha_n| < |\alpha_n+1|$ for $n \geqslant 1$.

Set

$$T^{(n)} = \lim_{k\to\infty} t_{n+1} \circ t_{n+2} \circ \cdots \circ t_{n+k}(0) \text{ for } n \geqslant 1,$$

provided these limits exists. Set

$$h_1 = 1,\ h_n = 1+\frac{a_n}{1+}\frac{a_{n-1}}{1}+\cdots+\frac{a_2}{1} \text{ for } n \geqslant 2.$$

In addition, we shall refer to a complex number, $\alpha \neq 0$, in the vicinity of the a_n's, and shall set $a = \mu(\mu+1)$, where $|\mu| < |\mu+1|$.

Theorem 2. *If*

(i) *there exists a fixed* $\eta > 0$ *such that* $|\alpha_n| + \eta < \dfrac{1}{5}$ *for* $n \geqslant 1$,

(ii) *there exists a complex number* $a \neq 0$ *such that* $|\mu| < \dfrac{1}{5}$,

(iii) $\sup\limits_{m \geqslant n} |a_m - a| < \epsilon_n < \dfrac{3|\mu|}{10(1+|\mu|^{-1})}$, $n \geqslant 1$,

then μ *is a converging factor of* (1) *and*

$$|T_n(\mu) - T| < \frac{25}{3}\epsilon_n |\mu|^{-2} . |T_n(0) - T| \text{ for } n \geqslant 1.$$

Proof. (1) converges, since $|\alpha_n| < \frac{1}{5}$ implies $|a_n| < \frac{1}{4}$, and the Worpitzky circle theorem (Wall, 1948) applies. The equation

$$\left| \frac{T - T_n(\mu)}{T - T_n(0)} \right| = \left| \frac{T^{(n)} - \mu}{T^{(n)}} \right| \cdot \left| \frac{h_n}{h_n + \mu} \right|, \ n \geqslant 1 \tag{3}$$

is easily obtained in Thron and Waadeland (1980). Let $\epsilon'_n = \frac{5}{3}\epsilon_n$. Straightforward manipulation of inequalities leads to

$$|\alpha_m - \mu| < \epsilon'_n, \text{ if } |a_m - a| < \epsilon_n, \ \ m \geqslant n \geqslant 1.$$

Set $\delta_n = \epsilon'_n[2 + |\mu|^{-1}(1+\epsilon'_n)]$, $n \geqslant 1$. It is easily seen that $\delta_n < |\mu|$ and that $|z-\mu| < \delta_n$ implies $|1+z| > 2|\mu|$. With these results the following lemma can be proven.

Lemma.

$$|z - \mu| < \delta_n \Longrightarrow |t_m(z) - \mu| < \delta_n, \ \ m \geqslant n \geqslant 1.$$

Proof of Lemma.

$$|t_m(z) - \mu)| = \left| \frac{\alpha_m(\alpha_m + 1)}{1+z} - \mu \right| \leqslant \frac{|\alpha_m - \mu|(1 + |\alpha_m + \mu|) + |\mu| \cdot |z - \mu|}{|1+z|}$$

$$< \frac{\epsilon'_n(1 + |\alpha_m| + |\mu|) + |\mu|\delta_n}{2|\mu|} \leqslant \frac{(1 + 2|\mu| + \epsilon'_n)\epsilon'_n + |\mu|\delta_n}{2|\mu|} = \delta_n.$$

Returning to the proof of theorem 2, the lemma gives

$$|T_{n+k}(\mu)-\mu| < \delta_n \quad \text{for} \quad n \geqslant 1, \quad k \geqslant 1. \tag{4}$$

Now, $\mu \notin I_n$ for $n \geqslant 1$, since the radius of I_n, $\sqrt{|a_n|} = \sqrt{|\alpha_n(\alpha_n+1)|}$ is less than $\frac{1}{2}-\rho$, for some $\rho \in \left(0, \frac{1}{4}\right)$, and $|\mu| < \frac{1}{5}$

Applying theorem 1,

$$\lim_{k\to\infty} T_{n+k}(\mu) = \lim_{k\to\infty} T_{n+k}(0) = T^{(n)}, \quad n \geqslant 1.$$

Hence, from (4),

$$|T^{(n)}-\mu| < \delta_n, \quad n \geqslant 1.$$

Thus, in (3),

$$\left| \frac{T^{(n)}-\mu}{T^{(n)}} \right| \leqslant \frac{\delta_n}{|\mu|-\delta_n}. \tag{5}$$

And, since the a_n's are in the Worpitzky circle, $|h_n| \geqslant \frac{1}{2}$(Wall 1948). Therefore, in (3),

$$\left| \frac{h_n}{h_n+\mu} \right| \leqslant \frac{5}{3}. \tag{6}$$

(5) and (6), incorporated in (3), give

$$\left| \frac{T_n(\mu)-T}{T_n^{(0)}-T} \right| < \epsilon'_n \frac{5(2|\mu|+1+\epsilon'_n)}{3|\mu|^2-3\epsilon'_n(2|\mu|+1+\epsilon'_n)}$$

$$< 5\epsilon'_n|\mu|^{-2}$$

$$= \frac{25}{3}\epsilon_n|\mu|^{-2}.$$

This concludes the proof of theorem 2.

Example. Consider $\frac{a_1}{1}+\frac{a_2}{1}+\cdots$, with $\left| a_n-\frac{1}{6} \right| < 10^{-k},\ k > 2$,

then $\mu = \frac{\sqrt{15}-3}{6}$, and theorem 2 gives

$$|T_n(\mu)-T| < 4\times 10^{2-k}.|T_n(0)-T|, \quad n \geqslant 1.$$

If ϵ_n is very much smaller than $|\mu|^2$, then the factor preceding $|T_n(0) - T|$ in the last line of theorem 2 is quite small. Consequently, the use of the converging factor μ improves the convergence of the continued fraction (1). In the special case $\lim \alpha_n = \mu$, theorem 2 guarantees that the use of the converging factor μ will result in an actual acceleration of the convergence of (1).

References

Ford, L. (1951) : *Automorphic Functions*, 2nd Ed., Chelsea, New York.

Gill, J. (1975) : The Use of Attractive Fixed Points in Accelerating the Convergence of Limit Periodic Continued Fractions, *Proc. Amer. Math. Soc.*, 47, 119.

Perron, O. (1957) : *Die Lehre von den Kettenbrüchen*, 3. Aufl. 2. Band, B. G. Teubner, Stuttgart.

Thron, W. and Waadeland, H. (1980) : Accelerating Convergence of Limit Periodic Continued Fractions $K(a_n/1)$. *Numer. Math.*, 34, 2, 155.

Waadeland, H. (1966) : A Convergence Property of Certain T-Fraction Expansions, *K. Norske Vidensk. Selsk. Skr.* 9.

Wall, H. (1948) : *Analytic Theory of Continued Fractions*, D. Van Nostrand, New York.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN COLORADO
PUEBLO, COLORADO-81001

Bull. Cal. Math. Soc. 75, 69–75 (1983)

STUDY OF TURBULENT BOUNDARY LAYERS ON POROUS SURFACE

M. A. Abdel-Gaid

(*Received* 28 *June* 1980 : *Revision received* 2 *June* 1981)

Turbulent flows of electrically conducting media have so far been investigated considerably less than turbulence in ordinary liquids or gases. Most theoretical and experimental papers (Vatazhin, Lyubimov and Regirer 1970; Branover and Tsinober 1970; Ievlev 1975) are mainly concerned with constant-property media.

A number of papers are devoted to study of internal structure during turbulent motion in a boundary-layer. A numerical solution for turbulent boundary-layer at a permeable surface in crossed magnetic and inhomogeneous electric fields (Starostin 1978) have been obtained. In this paper an approximate solution have been obtained for the flow of an incompressible, electrically conducting liquid between two parallel walls.

Basic equations and its solutions : Consider the flow of an incompressible, electrically conducting liquid in a magnetic field :

$$B_y = \frac{B_0}{2x},\ x \neq 0,\ B_z = 0,\ B_x = 0$$

and in a nonuniform external electrical field :

$$E_x = 0,\ E_y = E_z = f(y, z).$$

Assuming then the flow is described by the equation of boundary-layer theory, we write :

$$u\ \frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu_0\ \frac{\partial^2 u}{\partial y^2} + \frac{1}{\rho} F_x$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where

$$F_x = J_y B_z - J_z B_y. \tag{1}$$

By Chom's law for isotropic conductivity, the current density equals :

$$\vec{J} = \sigma(\vec{E} + \vec{u} \wedge \vec{B}).$$

It is assumed that E_z is a function of (x, y) :

$$E_z = J_{av}\,\frac{8}{\sigma\pi}\left(\frac{1}{1+y_1^2}-\frac{1}{1+(4-y_1)^2}+\frac{1}{1+(4+y_1)^2}\right) \qquad (2)$$

$$y_1 = \sqrt{\frac{u_e}{x\nu_0}}\;\frac{y}{H}$$

where J_{av} is the average current density on the surface with characteristics dimension H.

The expression for F_x is then :

$$F_x = -J_{av}\,\frac{8B_0}{2x\pi}\left(\frac{1}{1+y_1^2}-\frac{1}{1+(4-y_1)^2}+\frac{1}{1+(4+y_1)^2}\right)-\frac{\sigma u B_0^2}{4x^2}. \qquad (3)$$

The boundary conditions are :

$$y = 0,\; u = 0,\; v = v_w,\; y = H,\; u = u_e.$$

To solve equations (1) with boundary conditions (4) we try to obtain a similar solution for the problem, we take :

$$u = \frac{\partial\psi}{\partial y}, \qquad v = \frac{\partial\psi}{\partial x},$$

$$\eta = (\nu_0 x)^{-\frac{1}{2}}u^{\frac{1}{2}}\,\frac{y}{H}, \qquad \psi = (\nu_0 x)^{\frac{1}{2}}u^{\frac{1}{2}}f(\eta). \qquad (5)$$

where ψ is the stream function,

$$u = \frac{\partial\psi}{\partial y} = \frac{u_e}{H}f'(\eta),$$

$$v = \frac{\nu_0^{\frac{1}{2}}u_e^{\frac{1}{2}}}{2x^{\frac{1}{2}}}\,(\eta f''(\eta)-f(\eta)),$$

$$\frac{\partial u}{\partial y} = \frac{(\nu_0 x)^{-\frac{1}{2}}}{H^2}\,u_z^{-3/2}f''(\eta),$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{u_e^2}{\nu_0 x H^3}\,f''',\;\; \frac{\partial u}{\partial x} = -\frac{u_e}{2xH}\,\eta f''(\eta) \qquad (6)$$

Equation (1) take the form :

$$\frac{2f'''}{H} = -ff''+N_1f'+N_2I_1$$

where

$$N_1 = \frac{\sigma B_0{}^2 H}{\rho u_e}, \qquad N_2 = \frac{8B_0 J_{av} H^2}{\rho \pi u_0{}^2}$$

$$I_1 = \frac{1}{1+\eta^2} - \frac{1}{1+(4-\eta)^2} + \frac{1}{1+(4+\eta)^2}$$

The boundary conditions become :

$$\eta = 0 \qquad f = c, \qquad f' = 0$$

$$\eta = H, \quad f' = H \tag{8}$$

where c is a quantity related with suction velocity. Equation (7) can be solved by using the method of successive approximations. In the problem of boundary-layer we think that the best zero approximation may be taken in the form :

$$f_0 = c + \eta + \alpha e^{-\beta\eta} - \alpha \tag{9}$$

where α, β are two arbitrary constants chosen such that $f_e'(0) = 0$ and in the first approximation

$$f'_1(0) = 0 \quad \text{i.e.,} \quad \alpha\beta = 1.$$

Different approximations are obtained from the equation

$$\frac{2f_i'''}{H} = -f_{i-1}f'_{i-1} + N_1 f'_{i-1} + N_2 I_1, \quad i = 1, 3, 3, \ldots \tag{10}$$

where the l.h.s. of (10) is known as the expressions on the r.h.s. are all known; if we use (9) in (10) then the first approximation is given from :

$$\frac{2f_1'''}{H} = -f_0 f_0'' + N_1 f_0' + N_2 I_1 \tag{11}$$

under the boundary condition (8) the integration of (11) gives :

$$\frac{2f'_1}{H} = -\frac{1}{\beta}\left[c+\eta+\frac{1}{\beta}(N_1+1)\right]e^{-\beta\eta} - \frac{1}{4\beta^2}e^{-2\beta\eta} + \frac{1}{2}N\eta^2$$

$$+N_2\Big[\eta\tan^{-1}\eta - \frac{1}{2}\log(1+\eta^2) + \frac{1}{2}\log(1+(4-\eta)^2)$$

$$-(4-\eta)\tan^{-1}((4-\eta) + (4+\eta)\tan^{-1}(4+\eta) - \frac{1}{2}\log(1+(4+\eta)^2)\Big]$$

$$+\eta\left(c+\frac{1}{2\beta}+\frac{N_1}{\beta}+2N_1\tan^{-1}4\right) + \frac{c}{\beta} + \frac{5}{4\beta^2} + \frac{N_1}{\beta^2}. \tag{12}$$

$$\frac{2f_1}{H} = \frac{1}{\beta^2}\left[c+\frac{1}{\beta}(N_1+1)\right]e^{-\beta n}+\frac{1}{\beta^2}\left(\eta+\frac{1}{\beta}\right)e^{-\beta\eta}+\frac{1}{8\beta^3}e^{-2\beta\eta}$$

$$+\frac{1}{6}N_1\eta^3+N_2[\tfrac{1}{2}(\eta^2-1)\tan^{-1}\eta+\tfrac{1}{2}(8+\eta)+\tfrac{1}{2}\{(4-\eta)^2+1\}$$

$$\times\tan^{-1}(4-\eta)+\tfrac{1}{2}\{(4+\eta)^2-1\}\tan^{-1}(4+\eta)-\tfrac{1}{2}(4-\eta)\log(1+(4+\eta)^2)$$

$$-\tfrac{1}{2}(4+\eta)\log(1+(4-\eta)^2)-\tfrac{1}{2}\eta\log(1+\eta^2)]+\frac{\eta^2}{2}\left[c+\frac{1}{2\beta}+\frac{N_1}{\beta}\right.$$

$$\left.+2N_1\tan^{-1}4\right]+\left(\frac{c}{\beta}+\frac{5}{4\beta^2}+\frac{N_1}{\beta^2}\right)\eta+\frac{2c}{H}-\frac{c}{\beta^2}-\left(N_1+\frac{17}{8}\right)\frac{1}{\beta^3}$$

$$-N_2[4+16\tan^{-1}4-4\log 17]. \tag{13}$$

From (12) we can calculate the displacement thickness defined by

$$\delta = \int_0^H \left(1-\frac{f'_1}{H}\right)d\eta$$

$$= H-\frac{f_1(H)}{H}+\frac{f_1(0)}{H} \tag{14}$$

where

$$\frac{f'_1}{H}=\frac{u}{u_e} \qquad \text{from (6).}$$

After evaluating with the help of the above formulae, the results can be elaborated with the help of tables (1–3) and figures (1–3).

TABLE (1)

η	f'_1/H		
	$c = 0.1$	$c = 0.5$	$c = 1$
0	0	0	0
0.2	1.0693	1.1484	1.2471
0.4	2.0298	2.1901	2.3852
0.6	2.9156	3.1451	3.3141
0.8	3.7314	4.0364	4.4177
1	4.5066	4.8836	5.3548

(1) For constant N_1, N_2, β, the velocity increases when η and the suction parameter c increase, cf. table (1) and figure (1).

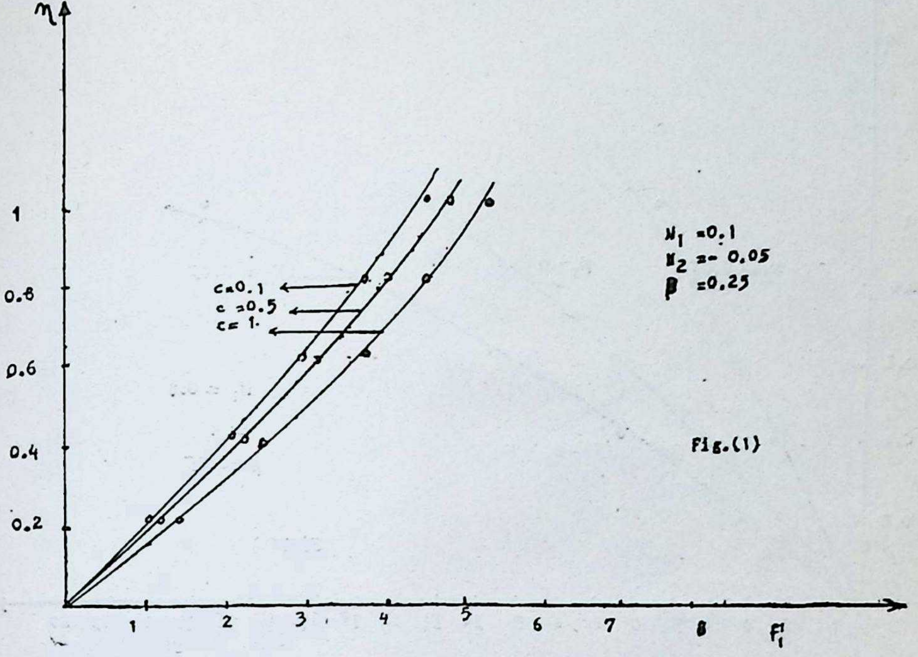
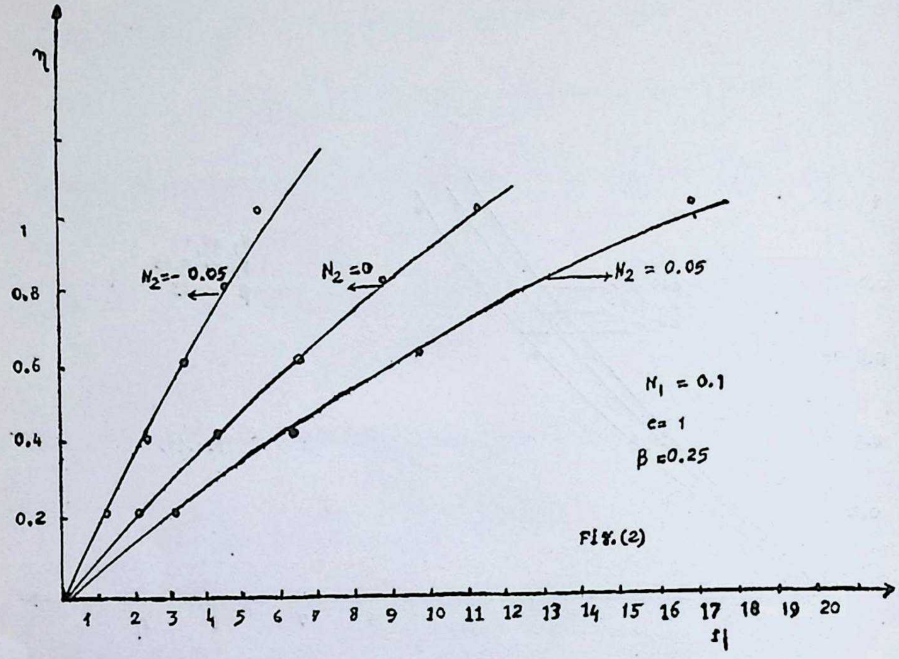


Fig.. 1

TABLE (2)

η	f'_1/H		
	$N_2 = -0.05$	$N_2 = 0$	$N_2 = 0.05$
0	0	0	0
0.2	1.2471	2.1968	3.1463
0.4	2.3852	4.3890	6.3928
0.6	3.3144	6.5767	9.7185
0.8	4.4177	8.7602	13.1027
1	5.3548	10.9397	16.5246

(2) The velocity at any point increases when the average current density (N_2) increases, cf. table (2), figure (2).



Fi2. 2

TABLE (3)

η	f'_1/H		
	$N_1 = 0.1$	$N_1 = 0.3$	$N_1 = 0.6$
0	0	0	0
0.2	2.1968	5.3954	10.1933
0.4	4.3890	10.7868	20.3824
0.6	6.5767	16.1732	30.5680
0.8	8.7602	21.5564	40.7508
1	10.9397	26.9361	50.9313

(3) The velocity at any point increases when the magnetic field (N_1) increases, cf. table (3), figure (3).

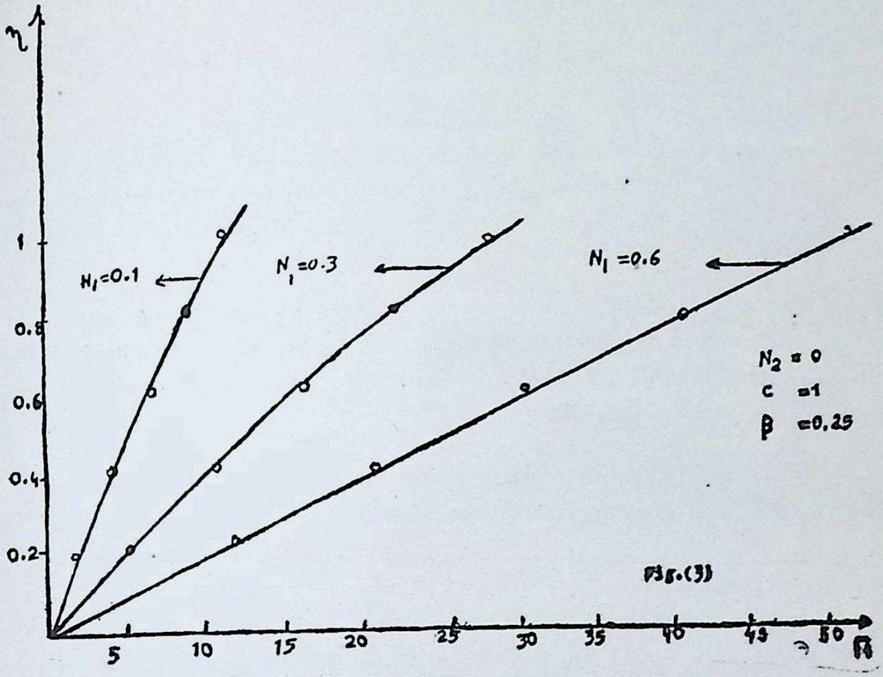


Fig 3,

References

Branover, G. G. and Tsinober, A. B. (1970) : Magnetohydrodynamics of incompressible Media (in Russian), Nauka, Moscow.

Ievlev, V. M. (1975) : Turbulent motion of high-temperature continuous media (in Russian), Nauka, Moscow.

Starostin, A. D. (1978) : Turbulent boundary-layers at a permeable surface in crossed magnetic and inhomogeneous electric fields. (Translated from Magnitnaya Gidrodinamika, 3, 41 July-September.

Vatazhin, A. B., Lyubimov, G. A. and Regirer, S. A. (1970) : Magnetohydrodynamic flows in channels (in Russian) Nauka, Moscow.

ASSIUT UNIVERSITY
EGYPT.

Bull. Cal. Math. Soc. 75, 77-81 (1983)

CONVERGENCE OF FOURIER COEFFICIENTS' SERIES FOR VECTOR VALUED FUNCTIONS

M. A. KANDIL

(*Received* 17 *July* 1980)

Abstract : This article is devoted to the convergence of Fourier coefficients' series for periodic vector valued functions via the modulus of continuity. It shows that for such functions with values in Hilbert space, the series are absolutely convergent. On the other hand, for values in weakly sequentially complete Banach space, the Fourier coefficients' series is strongly unconditionally convergernt. An example is given.

1. **Introduction**. This article is one of a series of studies carried by the author on Fourier series for vector valued functions with values in a Banach space, and of strong bounded variation. Here the convergence of the Fourier coefficients is studied via the modulus of continuity.

We denote by X a Banach space with the norm $||\cdot||$. We study a periodic X-valued function $x(t)$ with the real variable t and of period 2π. The Fourier coefficients of $x(t)$ are defined as

$$C_k = \frac{1}{2\pi} \int_0^{2\pi} x(t) \exp(-ikt)dt, \quad k = 0, \pm 1, \pm 2, \ldots$$

On analogy to real valued functions, we define here the modulus of continuity $\omega(x, h)$ for $x(t)$ by the following :

$$\omega(x, h) = \sup_{|t_1 - t_2| \leqslant h} ||x(t_1) - x(t_2)||$$

The integrated modulus of continuity is

$$\omega_p(x, h) = \sup_{|\eta| \leqslant h} \left(\int_0^{\pi} ||x(t+\eta) - x(t)||^p \, dt \right)^{1/p}.$$

We have to clearify relations between the values in Banach space of a vector valued function and the convergence of its Fourier coefficients' series. The function is of strong bounded variation and satisfies the condition :

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x, \frac{1}{n}\right)}}{n} < +\infty$$

2. THEOREM I. *If* : (1) $x(t)$ *is a periodic vector valued function of real variable* t *with period* 2π, *and with values in Hilbert space* H

(2) $x(t)$ *is* : (i) *Strongly continuous, and* (ii) *of strong bounded variation on* $[0, 2\pi]$

(3)
$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x, \frac{1}{n}\right)}}{n} < +\infty$$

then,

$$\sum_{k=-\infty}^{+\infty} ||C_k|| < +\infty.$$

Proof of the theorem I. $x(t)$ is strongly continuous and of strong bounded variation on $[0, 2\pi]$, hence we have (Kandil 1967).

$$x(t+h)-x(t) = \sum_{k=-\infty}^{+\infty} C_k(e^{ikh}-1)e^{ikt}.$$

Applying Parseval's identity (Kandil) one gets

$$\frac{1}{2\pi}\int_0^{2\pi} ||x(t+h)-x(t)||^2\,dx = \sum_{k=-\infty}^{+\infty} |e^{ikh}-1|^2 ||C_k||^2 = 4\sum_{k=-\infty}^{+\infty} ||C_k||^2 \sin^2 \frac{kh}{2}$$

So

$$\sum_{k=-\infty}^{\infty} ||C_k||^2 \sin^2\frac{kh}{2} \leqslant \frac{1}{2\pi}\,\omega^2{}_2(x, h).$$

Let $h = \frac{\pi}{2^{p+1}}$ the above equation becomes :

$$\sum_{k=2^{p-1}}^{2^p} ||C_k||^2 \sin^2\frac{k\pi}{2^{p+1}} \leqslant \frac{1}{2\pi}\,\omega^2{}_2(x, 2^{-p-1}\pi).$$

and thus

$$\sum_{k=2^{p-1}}^{2^p} ||C_k||^2 \leqslant \frac{1}{\pi}\,\omega^2{}_2(x, 2^{-p-1}\pi).$$

By Cauchy's inequality

$$\sum_{k=2^{p-1}}^{2^p} ||C_k||^2 \leqslant \left(\sum_{k=2^{p-1}}^{2^p} ||C_k||^2\, 2^{p-1}\right)^{\frac{1}{2}}$$

$$\leqslant \frac{2^{p/2}}{\sqrt{2\pi}}\,\omega_2(x, 2^{-p-1})$$

Then if,

$$\sum_{n=1}^{\infty} \frac{\omega_2\left(x, \frac{1}{n}\right)}{\sqrt{n}} < +\infty \tag{1}$$

i.e.,

$$\sum_{p=1}^{\infty} 2^{p/2}\omega_2(x, 2^{-p}) < +\infty$$

one gets,

$$\sum_{k=0}^{\infty} ||C_k|| < +\infty.$$

With the same procedure for negative values of k, we deduce that if (1) is valid then

$$\sum_{k=-\infty}^{+\infty} ||C_k|| < +\infty.$$

We deduce the relation between $\omega_2\left(x, \frac{1}{n}\right)$ and $\omega\left(x, \frac{1}{n}\right)$ when $x(t)$ is a vector valued function of strong bounded variation.

Let, $0 \leqslant \eta \leqslant h$

$$\int_0^{2\pi} \| x(t+\eta)-x(t-\eta)\|^2 dt < \max_{0\leq t\leq 2\pi} ||x(t+\eta)-x(t-\eta)|| \int_0^{2\pi} \| x(t+\eta)-x(t-\eta)|| dt.$$

So we have,

$$\omega^2{}_2(x, h) \leqslant 4\omega(x, h), \omega_1(x, h).$$

Since $x(t)$ is of strong bounded variation, we get

$$\omega_2(x, h) \leqslant A\sqrt{h\omega(x, t)} \qquad (A \text{ is a constant})$$

So if,

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x, \frac{1}{n}\right)}}{n} < +\infty$$

we get

$$\sum_{n=1}^{\infty} \frac{\omega_2\left(x, \frac{1}{n}\right)}{\sqrt{n}} < +\infty.$$

And this completes the proof.

3. THEOREM II. *If* (1) *$x(t)$ is a periodic vector valued function of real variable t, with period 2π and with values in weakly sequentially complete Banach space*

(2) *$x(t)$ is* : (i) *strongly continuous, and* (ii) *of strong bounded variation on* $[0, 2\pi]$

(3)
$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x, \frac{1}{n}\right)}}{n} < +\infty$$

then the series $\sum_{n=-\infty}^{+\infty} C_n$ *is strongly unconditionally convergent.*

Proof of theorem II. Let X^* is the conjugate space of X. Consider the numerical function $x^*x(t)$, where x^* is a linear functional in X^*. The Fourier coefficients of $x^*x(t)$. are x^*C_k $(k = 0, \pm 1, \pm 2, \ldots)$. Apply Perseval's identity to $x^*x(t)$, we get

$$4 \sum_{-\infty}^{+\infty} |x^*C_k|^2 \sin \frac{{}^2kh}{2} = \frac{1}{2\pi} \int_0^{2\pi} |x^*(x(t+h)-x(t)|^2 dt$$

$$< \frac{1}{2\pi} \int_0^{2\pi} \|x(t+h)-x(t)\|^2 dt.$$

Repeating the steps of the previous theorem, one gets

$$\sum_{\infty}^{\infty} |x^*C_k| < +\infty.$$

So, any subsequence of the partial sums of the series $\sum_{k=-\infty}^{+\infty} C_k$ is weakly convergent. Using the given condition that the space values is weakly sequentially complete Banach space, we deduce that the series $\sum_{k=-\infty}^{+\infty} C_k$ is weakly unconditional convergent.

Apply the fact that in Banach space, each weakly unconditional convergent series is strongly unconditional convergent (Hille 1974). Hence the theorem is proved.

4. **Example** In this section we introduce a $L(0, 2\pi)$ valued function defined on $[0, 2\pi]$ and satisfying *all the requirements of theorem II*. We show that the Fourier coefficients' series is not absolutely convergent.

Take $x(t)$, defined as follows

$$x(t) = \begin{cases} 1 & 0 \leqslant \sigma \leqslant t \\ 0 & t < \sigma \leqslant 2\pi. \end{cases}$$

It is clear that

$$||x(t+\eta)-x(t)|| = \int_t^{t+n} dt = \eta.$$

So, $x(t)$ is strongly continuous and of strong bounded variation on $|0, 2\pi|$. Also

$$\omega\left(x, \frac{1}{n}\right) = \sup_{|h| < 1/n} ||x(t+h)-x(t)|| = \frac{1}{n}.$$

So

$$\sum_{n=1}^{\infty} \frac{\sqrt{\omega\left(x, \frac{1}{n}\right)}}{n} < +\infty.$$

It is easy to calculate its Fourier series coefficients with the results:

$$C_0 = \left(1-\frac{\sigma}{2\pi}\right), \qquad C_k = \frac{i}{2\pi k}\{(1-\cos k\sigma)+i\sin k\sigma\}$$

where,

$$||C_0|| = \pi$$

$$||C|| = \int_0^{2\pi} |C_k| \, d\sigma = \int_0^{2\pi} \left|\frac{\sin\frac{k}{2}\sigma}{\pi k}\right| d\sigma = \frac{4}{\pi|k|}$$

and hence Fourier coefficients' series is not absolutely convergent.

This means that conditions of theorem II are insufficient for the absolute convergence of Fourier coefficients' series.

References

Bari, N. K. (1961, 1964): Trigonometric series, Fizmatgiz, Moscow, English transl., Macmillan. New York; Pergamon Press, Oxford, MR 23 $\neq$ A3411; Mr. 30 $\neq$ 1347.

Hille, E. and Phillips, R. S. (1974): Functional analysis and Semi-groups, *Amer. Math. Soc. Publ.*, **31**, rev. ed., *Amer. Math. Soc.*, Providence R.I.

Kandil, M. A. (1967): Poisson's summation formulae for vector valued functions, *Vestnik*. Moscow State University, 4, 38.

——— (1967): The Wiener-Paley theorem and other questions in the theory of abstract function *Math. Sbornik* Tom **73** (115), 3, 306; English transl., *Math. USSR Sbornik* **2**, 3, 271.

Kandil, M. A. (1979): Vector valued functions of strong bounded variation, *Proc. Math. Phys. Soc.* Egypt. To be published in Vol. 48, Juyl 1979.

——— (1979): Fourier series of multivariable vector valued functions of strong bounded variation, *Proc. Math. Phys. Soc.* To be published in, 48, Juyl 1979.

———: Parseval's identity and Banach spaces, accepted for publication in *Bull. Faculty of Sc.* University of Cairo, Egypt.

Wik, I. (1972): Criteria for absolute convergence of Fourier series of functions of bounded variation, *Trans. Am. Soc.* **163**, 1.

Zygmund, A. (1959): Trigonometrical series, 2nd rev. ed., Cambridge Univ. Press, New York, Mr $\neq$ 6498.

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ASSIUT,
ASSIUT, EGYPT

Bull. Cal. Math. Soc. **75**, 83-93 (1983)

GRAPH TOPOLOGIES ON A SET

TARIT KUMAR MAZUMDER

(*Received 6 November 1978*)

Introduction. In this paper an attempt has been made to ascribe a topology on a Graph (Claude Berge), it has been possible to connect different ideas with the same name of Graph Theory and Topology e.g. connectedness [Theorem-8]; a Graph with the ascribed topology has been termed a Graph Topological space; a representation theorem for such a space is also formulated (Theorem 11).

§ 1. According to Claude Berge Page-5, a Graph is a non-empty set X, together with a function f and is denoted by the ordered pair (X, f).

Definition-1. Let (X, f) be a graph. For each subset A of X, let us define

$$m(A)=A \cup f(A).$$

Then, the collection,

$$\tau=\{A/A \subseteq X \text{ and } m(A)=A\}$$

defines a topology on X, call it the *Graph topology* on X induced by f. The ordered triple $(X, f; \tau)$ will stand for a Graph topological space induced by the Graph (X, f).

If $(X, f; \tau)$ be a Graph topological Space, the following remarks can be made easily :-

Remark-1. τ is closed with arbitrary intersection of open sets.

Remark-2. The fixed points of f are only the isolated points of $(X, f; \tau)$ and conversely.

Remark-3. τ is discrete iff f is the identity mapping.

Note-4. In the Graph topological space $(X, f; \tau)$, define, for each $x \varepsilon X$,

$$0(x)=\bigcap\{G/G \varepsilon \tau \;\&\; x \varepsilon G\} \tag{A}$$

Then by Remark-2, the collection $\{0(x)/x \varepsilon X\}$ forms an open base for τ. Now, define $x \leqslant y$ iff $0(x) \supseteq 0(y)$ for each pair $(x, y) \varepsilon X \times X$. Certainly, the relation '$\leqslant$' is reflexive and transitive. Moreover, it is antisymmetric iff τ is T_0. If τ is essentially T_1, it can be reduced to an equivalance relation.

Also for each $x \varepsilon X$, $O(x)=\{y/y \geqslant x\}$. This at once gives that τ is T_1 iff it is discrete as from (*A*) above $O(x)=(x)$ for each $x \varepsilon X$. Thus, one can have,

Remark-5. X can be preordered by '$\geqslant$' in such a way that the collection $\{O(x)/x \varepsilon X\}$ forms an open base for τ; where $O(x)=\{y/y \geqslant x\}$.

§ 2. Examples of Graph topological spaces

Example-1. Let X be an infinite set. Let $X= \bigcup_{n \varepsilon N} A_n$; N=the set of all natural numbers.

Such that $A_n \neq \phi$, for any $n \varepsilon N$ & $A_n \cap A_m=\phi$ if $m \neq n$ and $m, n \varepsilon N$. Now, for each $n \varepsilon N$, select $y_n \varepsilon A_n$, and fix it.

Let us define $f: X \rightarrow X$ by $f(x)=y_{n+1}$ iff $x \varepsilon A_n$.

Let $(X, f; \tau)$ be the Graph topological space induced by (X, f). The following properties of $(X, f; \tau)$ readily follow :

Property-1. Any non-null open set of $(X, f; \tau)$ has a non-null intersection with the set, $Y=\{y_1, y_2, \dots, y_n, \dots\}$ and hence $(X, f; \tau)$ is separable (and therefore satisfies countable chain condition).

Property-2. Any $G \subseteq Y$ is open iff $y_i \varepsilon G \Rightarrow y_j \varepsilon G$ for all $j > i$ in N and hence Y is open. [Proof is made on the induction of i].

Property-3. There exists no pair of disjoint non-empty open sets and hence the space is hyper-connected.

Property-4. For each $x \varepsilon X$, let Y_x be the subset of Y such that it contains all those Y_n satisfying $n > m$ where $x \varepsilon A_m$. Then

i) Y_x is unique and open for each $x \varepsilon X$

ii) $(x) \cup Y_x$ is open $[\because f(x) \varepsilon Y_k]$ and singly forms a local base at x.

iii) The collection $\{(x) \cup Y_k / x \varepsilon X\}$ forms an irreducible base for $(X, f; \tau)$

Property-5. $(X, f; \tau)$ is 1°-countable and it is not 2°-countable (even Lindeloff) if X is uncountable.

Note-1. If one defines $f(x)=Y_n$ iff $x \varepsilon A_n$ instead of what in Example-1, the Graph topological space thus induced by f will be no-longer hyper-connected. However it will be separable (and hence satisfy countable chain condition), 1°-countable but it is not 2°-countable (even Lindelöff) if X is uncountable.

Example-2. Let N be the set of natural numbers and $f: N \rightarrow N$ be defined by $f(n)=m$, where $(m, n) \varepsilon N \times N$ and m is the Ist even integer which is either equal to n or just immediate to n.

The local base at any $n \in N$, for the Graph topological space $(N, f\,;\,\tau)$ is a single subset of N containing singly the natural number n or the pair of natural number $(n, f(n))$ accordingly n is even or odd. This space $(N, f\,;\,\tau)$ is therefore 2°-countable.

Note-2. In each of the above example f is taken single-valued and non-injective.

Example-3. Let Z be the set of all integers endowed with the topology $\tau = \{B_n\}_{n \in N}$ where $B_n = \{m/m \geqslant n \text{ and } m \in Z\}$. Then τ is a Graph topology on Z, induced by the function $f: Z \to Z$ defined by $f(n) = n+1$ $n \in Z$. Here (Z, τ) is certainly 2°-countable & hyperconnected and σ-compact.

Let the Graph topology thus constructed be named the *upper order topology on Z.*

§ 3. Properties of Graph Topology

Section-1. Let $(X, f\,;\,\tau)$ be a Graph topological space. For each $x \in X$, define,

$$B_x = \{(x) \cup f(x) \cup f^2(x) \cup \dots \cup f^n(x) \cup \dots\}$$

where $f^n(x)$, $n \in N$ has its usual meaning. Then one can have the following results :

Result-1. For each $x \in X$; B_x is open.

Proof. In fact, $f(B_x) \subseteq B_x$ for each $x \in X$.

Result-2. If G is a non-null open set then for each $x \in G$, $B_x \subseteq G$ and hence $G = \bigcup\limits_{x \in G} B_x$.

Proof. G is open $\Rightarrow f(G) \subseteq G$ and hence for any $x \in G$, $f^n(x) \subseteq G$ for any $n \in N$.

Result-3. For each $x \in X$, B_x singly constitute a local base at x and hence the space is 1°-countable.

Result-4. The collection $\mathcal{B} = \{B_x / x \in x\}$ forms an irreducible open base for the Graph topology.

Note-5. According to Claude Berge [*Topological Spaces,* page-23,] the transitive closure of $T: X \to X$ is given by

$$\hat{T}(x) = \{(x) \cup Tx \cup T^2x \cup \dots\}$$

The correspondence $A \to \hat{T}(A)$ $(A \subseteq X)$ is a closure operator on X. It is noticed here that, the topology defined on X by taking $A \subseteq X$ closed iff $A = \hat{T}(A)$ is a sort of dual to the Graph topology on X [for closed sets and open sets satisfy arbitrary intersection and union property.]

Result-6. If $f(x)$ contains atmost a countable number of points for each $x \in X$, B_x is countable for each $x \in X$.

Result-7. If f is single-valued and injective

$$B_x \cap B_y \neq \phi \Rightarrow \text{either } B_x \subseteq B_y \text{ or } B_y \subseteq B_x.$$

Proof. $B_x \cap B_y \neq \phi$ implies the existence of $m, n \varepsilon N$ such that $f^m(x) = f^n(y)$. As f is injective it follows that,

i) $B_x = B_y$ if $m = n$

ii) $B_x \subseteq B_y$ if $m \leqslant n$

iii) $B_x \supseteq B_y$ if $m \geqslant n$.

THEOREM 8. *Let $(X, f\,;\, \tau)$ be a Graph topological space. Then for each $x \varepsilon X$, the component $C(x)$ of x is given by the formula :*

$$C(x) = \bigcup_{r=1}^{\infty} \bigcup_{\substack{p_1, p_2, \dots, p_r = 0 \\ q_1, q_2, \dots, q_r = 0}}^{\infty} f^{-p_1} 0 f^{q_1} 0 \dots 0 f^{-p_r} 0 f^{q_r}(x).$$

Then, further, $C(x)$, for each $x \varepsilon X$, is open and closed and hence $(X, f\,;\, \tau)$ is locally connected.

Proof. Let us set,

$$G = \bigcup_{r=1}^{\infty} \bigcup_{\substack{p_1, p_2, \dots, p_r = 0 \\ q_1, q_2, \dots, q_r = 0}}^{\infty} f^{-p_1} 0 f^{q_1} 0 \dots 0 f^{-p_r} 0 f^{q_r}(x)$$

The proof is made according to the following steps.

Step—I. G is open.

In fact, for any $y \varepsilon G$.

$$y \varepsilon f^{-p_1} 0 f^{q_1} 0 \dots 0 f^{-p_r} 0 f^{q_r} \text{ for some } r \geqslant 1$$

Then, for any + ve integer n,

$$f^n(y) \subseteq f^n 0(f^{-p_1} 0 f^{q_1} 0 \dots 0 f^{-p_r} 0 f^{q_r}(x)) \subseteq G$$

$$\Rightarrow B_y \subseteq G \Rightarrow \text{Step—I.}$$

Step—II. G is closed

In fact, let $y \notin G$ and if possible, $B_y \cap G \neq \phi$

$\Rightarrow$ There exists an integer $r \geqslant 1$ for which

(1) $z \varepsilon f^r(y)$

and also there exists integer $r_1 \geqslant 1$ for which

$$(2) \quad z \varepsilon f^{-p_1} 0 f^{a_1} 0 \dots 0 f^{-p_{r_1}} 0 f^{q_{r_1}}(x).$$

From (1) and (2), it follows that,

$$y \varepsilon f^{-(r+p_1)} 0 f^{q_1} 0 \dots 0 f^{-p_{r_1}} 0 f^{a_{r_1}}(x) \subseteq G$$

which is a contradiction. Hence Step II follows.

Step—III. *The set* $\bigcup\limits_{\substack{p_1=0\\q_1=0}}^{\infty} f^{-p_1} 0 f^{q_1}(x) \subseteq C(x)$.

In fact, proof of Step—III is made by induction on p_1. when $p_1=0$, certainly,

$$\bigcup_{q_1=0}^{\infty} f^{q_1}(x) \subseteq C(x) \text{ as } \bigcup_{q_1=0}^{\infty} f^{a_1}(x)$$

is a connected set containing x.

Let us now assume that

$$U = \bigcup_{p_1=0}^{n} \bigcup_{q_1=0}^{\infty} f^{-p_1} 0 f^{a_1}(x) \subseteq C(x)$$

$$\text{Let} \quad V = \bigcup_{p_1=0}^{n+1} \bigcup_{a_1=0}^{\infty} f^{-p_1} 0 f^{a_1}(x).$$

Let $z \varepsilon V$ be arbitrary. Then, $z \varepsilon f^{-p_1} 0 f^{q_1}(x)$ for some integer p_1, q_1 such that $0 \leqslant p_1 \leqslant n+1$; $0 \leqslant q_1 < \infty$

Then, there exists $a+ve$ integer r, for which

$$f^{r}(z) \subseteq f^{r} 0 \left(f^{-p_1} 0 f^{q_1}(x) \right) \subseteq U \Rightarrow z \varepsilon \bar{U} \subseteq C(x)$$

[$\because U \subseteq C(x)$ and $C(x)$ being component is closed $\Rightarrow V \subseteq C(x)$.

Thus the induction on p_1 is complete and this proves Step—III.

Step—IV. If for some integer $n \geqslant 1$

$$H = \bigcup_{r=1}^{n} \bigcup_{\substack{p_1, p_2, \dots, p_r=0\\ q_1, q_2, \dots, q_r=0}}^{\infty} f^{-p_1} 0 f^{q_1} 0 \dots 0 f^{-p_r} 0 f^{a_r}(x) \subseteq C(x).$$

Then also,

$$K = \bigcup_{r=1}^{n+1} \bigcup_{\substack{p_1, p_2, \ldots, p_r = 0 \\ q_1, q_2, \ldots, q_r = 0}}^{\infty} f^{-p_1} 0 f^{q_1} 0 \ldots 0 f^{-p_r} 0 f^{q_r}(x) \subseteq C(x)$$

In fact, $z \varepsilon K$ be arbitrary. Then, $z \varepsilon f^{-p_1} 0 f^{q_1} 0 \ldots 0 f^{-p_r} 0 f^{q_r}(x)$ for some integer $r, 1 \leqslant r \leqslant n+1$.

Then, for some + ve integer t,

$$f^{t}(z) \subseteq f^{t} 0 (f^{-p_1} 0 f^{q_1} 0 \ldots 0 f^{-p_r} 0 f^{q_r}(x)) \subseteq H.$$

$$B_z \cap H \neq \phi \Rightarrow z \varepsilon \bar{H} \subseteq C(x) \text{ [as } \bar{H} \subseteq C(x) \text{ and } C(x) \text{ is closed.]}$$

This proves Step—IV.

Conclusion: By Step—III and IV it follows that $G \subseteq C(x)$, by the induction on r. Again $C(x) \subseteq G$ by Step—I and II. Hence $G = C(x)$. This proves Theorem—8.

Note—9. The components are given by the equivalence relation generated by the relation $P : (x, y) \varepsilon P$ iff $y \varepsilon f^{r}(x)$ for some $r \varepsilon z$

[An equivalence relation generated by a given relation P means the intersection of all equivalence relation containing P]

This shows that the idea of connectedness in the Graph Theory coincides with the same in topology.

Note—10: If f is single-valued, the components in Theorem—8, take the most elegant form : $C(x) = \bigcup_{n=-\infty}^{\infty} f^{n}(x)$, these components are given by the equivalence relation $P \subseteq X \times X$, defined by the rule :— $(x, y) \varepsilon P$ iff $x = f^{r}(y)$ for some $r \varepsilon Z$.

Representation Theorem: The following theorem gives a sort of representation of a Graph topological space. Here, the following notations should be used.

(a) For each index α, Z_{α} means a copy of Z (the set of integers) with the Upper Order topology (Example—3).

(b) For each index n_x, $\bar{Z}_{n_x}$ is the set $\{n / n \varepsilon Z, n > -n_x\}$ with its relative topology in Z_{α}.

(c) For each index α, $Z_{n_\alpha}^{\nabla}$ is the set of integers given by $\{1, 2, 3, \ldots, n_{\alpha} - 1\}$ with indiscrete topology.

THEOREM—11. *If f is single-valued, the Graph topological space $(X, f; \tau)$ is topologically homeomorphic to*

$$\underset{\alpha \varepsilon \Lambda_1}{\vee} Z_\alpha \oplus \underset{\alpha \varepsilon \Lambda_2}{\vee} Z_{n\alpha} \oplus \underset{n_x \varepsilon \Lambda_3}{\vee} \overline{Z}_{nx}$$

where $\vee, \oplus$ *denote the topological sum and the cardinality of* $\Lambda_1 \cup \Lambda_2 \cup \Lambda_3$ *is the same as that of the quotient set of X given by the equivalence relation* $P \subseteq X \times X : (x, y) \varepsilon P$ iff $x \varepsilon f^r(y)$ *for some* $r \varepsilon Z$.

Moreover,

i) $\Lambda_2 \cup \Lambda_3 = \phi$ if f is bijective and $f^n (n \geqslant 1)$ has no fixed pt.

ii) $\Lambda_2 \neq \phi$ iff is bijective and $f^n (n \geqslant 1)$ has fixed points.

iii) $\Lambda_3 \neq \phi$ iff is not surjective (but injective).

Proof. We need consider the following cases :

Case—1. f is bijective and f^n has no fixed pts. for $n \geqslant 1$. In this case, X with its Graph topology is certainly topologically homeomorphic to $\underset{\alpha \varepsilon \Delta_1}{\vee} Z_\alpha$ This at once follows from the fact that any component $C(x)$ of $x[x \varepsilon X]$ is given by $\bigcup_{n=-\infty}^{\infty} f^n(x)$ which is open and closed [vide Th. 8].

Case—2. f is bijective and f^n has fixed pts. for $n \geqslant 1$. In this case, let

$$F = \{x / x \varepsilon X \ \& \text{ for some } n \geqslant 1 \, ; f^n(x) = x\}$$

In F, define the equivalence relation given in the Theorem. Let $(x_\alpha)_{\alpha \varepsilon \Lambda_2}$ be the set of all equivalence classes. Certainly $\Lambda_2 \neq \phi$ as $F \neq \phi$. Let, for each $\alpha \varepsilon \Lambda_2$, n_α be such an integer that, $n_\alpha \geqslant 1$ and $f^{n_\alpha}(x_\alpha) = x$ and n_α is least such integer.

Then $(x_\alpha) = \{x_\alpha, f(x_\alpha), \ldots, f^{n_\alpha - 1}(x_\alpha)\}$. Tnen, X with Graph topology is topologically homeomorphic to

$$\underset{\alpha \varepsilon \Delta_1}{\vee} Z_\alpha \oplus \underset{\alpha \varepsilon \Delta_2}{\vee} Z_{n\alpha}$$

Case-3. f is not surjective, but injective.

In this case, let for each $x \varepsilon X$, n_x is the least +ve integer, such that, $f^{-n_x}(x) = \phi$. If for some $x \varepsilon X$, $f^{-n}(x) \neq \phi$ for any integer $n \geqslant 0$, we set $n_x = \infty$ and $Z_{n_x} = \overline{Z}_\alpha$ and then,

$$X \cong \underset{\alpha \varepsilon \Delta_1}{V} Z_\alpha \oplus \underset{\alpha \varepsilon \Delta_2}{V} Z_{n\alpha} \oplus \underset{n_x \varepsilon \Lambda_3}{\overrightarrow{Z}}$$

This completes the proof.

In view of Note-10, it follows that in the Graph topological space $(X, f; \tau)$, the components are almost countable if f is single-valued and injective. Hence it, at once, follows from Theorem-11, that,

THEOREM-12. *If $f: X \to X$ is single-valued and injective, the following are equivalent in the Graph Topological Space $(X, f; \tau)$:*

(*a*) *X is a countable set*

(*b*) *$(X, f; \tau)$ is second countable space*

(*c*) *$(X, f; \tau)$ is heriditarily Lindeloff Space*

(*d*) *$(X, f; \tau)$ is heriditarily Separable Space*

(*e*) *$(X, f; \tau)$ satisfies countable chain condition.*

Note-13. The requirement that f is injective is only for the proof (e)⇒(a). The essentiality of taking f injective is proved by example (I) and (II).

Note-14. Theorem-12 is also true if f is a multivalued function and such that for each $x \varepsilon X, f(x)$ and $f^{-1}(x)$ are countable subsets of x. In fact, it is from theorem-8, that the components of $x (x \varepsilon X)$ is given by

$$\bigcup_{r=1}^{\infty} \bigcup_{\substack{p_1 p_2, \ldots, p_r = 0 \\ q_1, q_2, \ldots, q_r = 0}}^{\infty} f^{-p_1} 0 f^{q_1} 0 \ldots 0 f^{-p_r} 0 f^{q_r} (x)$$

which is open and closed and countable subset of x.

Result-15. The Graph Topological Space $(X, f; \tau)$ is T_0 iff f is injective and f^n has no fixed point for $n > 1$. [This at once follows from Theorem-11.]

Result-16. The function $f: (X, f; \tau) \rightarrow (X, f; \tau)$ is a topological morphism (i.e. open and continuous).

Note-17. Let (X, f) be a graph. Let $U \subset X \times X$ be defined by the collection of all arcs of the Graph, i.e.

$$U = \{(x, f(x)) / x \varepsilon X\}.$$

Define : a mapping $l: X \rightarrow X \times X.$ by $l(x) = (x, f(x))$

By result-16, it follows that, if (x_α) be a net in X and $(x_\alpha) \rightarrow x \varepsilon X$, then $(l(x_\alpha)) \rightarrow l(x)$ in $X \times X$ where $X \times X$ is given its product topology. This shows that l is also topological morphism. Moreover, as $l(X) = U$, it follows that $l: X \rightarrow U$ is a topological homeomorphism and U is closed & open in $X \times X$. According to Claude Berge (*Thery of Graphs etc.* page-6), Subgraph of a graph (X, f) is defined to be a Graph of the form (A, f_A) where $A \subset X$ and in which the function f_A is defined by

$$f_A(x) = f(x) \cap A.$$

This leads to say that

Result-18. If (A, f_A) is a Subgraph of the Graph (X, f), the Graph topological space (A, f_A, τ_{f_A}) is the relative topology of $A \subset (X, f, \tau_f)$.

Again, according to Claude Berge (*Theory of Graphs etc.* P-23), the Graph (X, f) is called the product of n graph (X_i, f_i), $i = 1, 2, \dots n$, iff

$$X = X_1 \times X_2 \times \dots \times X_n$$

and
$$f(x_1, x_2, \dots x_n) = f_1(x_1) \times f_2(x_2) \times \dots \times f_n(x_n)$$

This again leads to say

Result-19. If $(X_i, f_i\ ;\ \tau_i)_{i=1}^{n}$ be a collection of Graph topological spaces (where τ_i is induced by (X_i, f_i) ; $i = 1, 2, \dots, n$), the product Graph $(X, f) = \overset{n}{\underset{i=1}{X}}\ (X_i, f_i)$ has its graph topology equivalent to the product topology.

Section-2. Compactness in the Graph topological Space

Result-20. For each $x \varepsilon X$, B_x is compact and hence, the Graph topological space $(X, f\ ;\ \tau)$ is Locally Compact.

Proof. If, for an index set I, $\{G_\alpha / \alpha \varepsilon \mathrm{I}\}$ be any open covering of B_x, there exists an $\alpha \varepsilon \mathrm{I}$, for which $x \varepsilon G_\alpha$. As B_x solely constitutes a local base at x, $x \varepsilon B_x \subseteq G$. Hence B_x is compact. This completes the proof.

Result-21. Let $(X, f\ ;\ \tau)$ be any Graph topological space. Then $A \subseteq X$ is compact iff there exists a finite number of points $x_1, x_2, \dots, x_n$ in A, such that $A \subseteq \bigcup_{i=1}^{n} B_{x_i}$.

Proof. If $A \subseteq X$ is compact, the 'if part' follows immediately by considering the covering $\{B_x / x \varepsilon A\}$. Conversely, if the condition holds, let $\{G_\alpha / \alpha \varepsilon \mathrm{I}\}$; I be any index set, be any open covering of X. Then for $i = 1, 2, \dots, n$; there exists $\alpha_i \varepsilon \mathrm{I}$ for which $x_i \varepsilon G_{\alpha i}$. Correspondingly, $A \subseteq \bigcup_{i=1}^{n} B_{x_i} \subseteq \bigcup_{i=1}^{n} G$. This shows A is compact. This completes the proof.

Result-22. If in $(X, f\ ;\ \tau)$; $B_y \cap B_z = \phi$ for some $y, z \varepsilon X$; then, for each $x \varepsilon X$, B_x^{-1} can intersect almost one of B_y and B_z.

Proof. If possible, let there exist a $x \varepsilon X$, such that $B_x \cap B_y \neq \phi$; $B_x \cap B_z \neq \phi$. Then there exist x_1 & x_2 for which

$$\left.\begin{array}{l} x_1 = f^{m_1}(x) = f^{n_1}(y) \\ x_2 = f^{m_2}(x) = f^{n_2}(y) \end{array}\right\} \begin{array}{l} \text{for some +ve integers} \\ m_1, m_2, n_1, n_2. \end{array}$$

If $m = \max [m_1, m_2]$.

We see $f^m(x) \varepsilon B_y \cap B_z$ which is a contradiction. This completes the proof.

Definition-23. A topological space is said to satisfy the *finite chain condition* iff any collection of pair-wise disjoint open sets is finite.

Result-24. If a Graph topological space $(X, f; \tau)$ is compact, it satisfies finite chain condition.

Proof. Let us assume that $(X, f; \tau)$ be compact. Then by Result-21, there exists a finite set of points $x_1, x_2, \ldots, x_n$ in X such that $X = \bigcup_{i=1}^{n} B_{x_i}$.

Now, if $(X, f; \tau)$ does not satisfy the finite chain condition, there exists an infinite collection $\{B_{x_\alpha} / \alpha \varepsilon \mathrm{I}, x_\alpha \varepsilon X\}$ which is pair-wise disjoint. This shows that there exists an i, $i \leqslant i \leqslant n$, for which, B_{x_i} intersects more than one B_{x_α} for $\alpha \varepsilon \mathrm{I}$. This is a contradiction due to Result-22.

This completes the proof.

Note-25. The converse of the above result is not true, as it can be seen by the example-3, Art. 2. In fact, the set Z with upper order topology satisfies finite chain condition, but it is not compact, as the covering $\bigcup_{n=1}^{\infty} B_{-n}$ where $B_{-n} = \{x / x \varepsilon Z \ \& \ x \geqslant -n\}$ has no finite subcovering.

Definition—26. A locally compact space is σ-compact iff it is expressible as the union of a countable number of compact spaces.

THEOREM—27. *In a Graph topological space $(X, f; \tau)$ where $f(x)$ and $f^{-1}(x)$ are countable for each $x \varepsilon X$, each of the following are equivalent*

(*a*) *X is a countable set*

(*b*) *$(X, f; \tau)$ is Lindelöff*

(*c*) *$(X, f; \tau)$ is σ-compact*

Moreover, each of the above follows if $(X, f\,;\,\tau)$ *is compact.*

Proof. $(a) \Rightarrow (b)$ is obvious as X is countable implies $(X, f\,;\,\tau)$ is second countable space [$\because \{B_x/x \,\varepsilon\, X\}$ is countable.] To prove $(b) \Longrightarrow (c)$, consider the covering of X by $\{B_x/x \,\varepsilon\, X\}$. By (b), it therefore follows that $X = \bigcup\limits_{i \,\varepsilon\, N} B_{x_i}$ and hence (c) is proved by Result—20. To prove $(c) \Longrightarrow (a)$, it is known [James Dugundji [*Topology*, page—241], that (c) implies that $(X, f\,;\,\tau)$ is Lindelöff. Hence considering the covering $\{B_x/x \,\varepsilon\, X\}$, X is countable. To prove the second part of the theorem we need only show that if $(X, f\,;\,\tau)$ is compact, X is countable set ; which is obvious as by Result 21, $X = \bigcup\limits_{i=1}^{n} B_{x_i}$. This completes the proof.

Acknowledgement : I feel deeply grateful to Dr. T. K. Mukherjee of Jadavpur University for his kind help in the preparation of the paper. I am also thankful to Dr. S. Ganguly of Calcutta University for his supervision.

References

Claude Berge : Theory of Graphs and its application. Translated by Alison Doig.

Claude Berge : Topological Spaces. Translated by E. M. Paterson.

James Dugundji : Topology.

JR. RESEARCH FELLOW (UGC)
DEPT. OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA, INDIA.

Bull. Cal. Math. Soc., 75, 95-99 (1983)

NON-LINEAR COMPUTATIONAL MODEL OF A TIDAL RIVER UNDER THE ACTION OF WIND-STRESS AND UPLAND FLOW

ANADI KUMAR CHATTERJEE

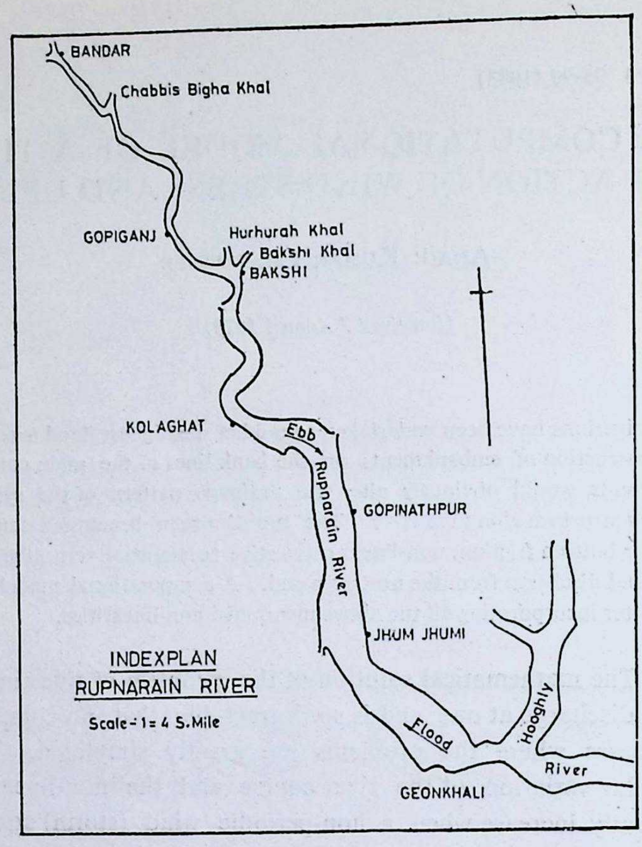
(*Received 7 August 1978*)

Abstract : Investigations have been undertaken to predict the highest flood levels along the Rupnarain River System for the construction of embankments on the bank lines of the main course as well as its tributaries. These embankments would obviously alter the drainage pattern of the system. It is therefore essential to study the flow structure along the river. The problem then becomes a non-linear problem due to the presence of non-linear bottom friction, non-linear convective acceleration term, the non-linear Wind-stress distribution and the upland discharge from the upstream end. A computational model is developed using the high speed digital computer incorporating all the above mentioned non-linearities.

Introduction : The mathematical solution of the problem of tides in a shallow irregular channel with upland discharge at one end is so intractable that no complete solution is yet possible except for cases where the problems are greatly simplified. The difficulties are entirely due to irregular variation of the river course and the non-linearity of the system. The complexities greatly increase when a non-periodic wind (storm) stress is superposed on the periodic tides. Such problems can be most accurately investigated by using the finite difference methods of solution.

This paper deals with a computational model (using the finite difference method) of the river Rupnarain (Fig. 1) under the combined action of abnormal upland discharge from the upstream end and a non-periodic wind stress (associated with cyclonic storms). The importance of the Rupnarain as the principal drainage artery of the Western Districts of West Bengal and also as the main tributary of the Hooghly (its influence on the navigable channel leading to the port of Calcutta and Haldia) is very great. Extensive study of the river Rupnarain for the dry season conditions i. e. without any upland discharge has been done by the author (Chatterjee 1972, p. 151, 1972, 1974 & 1973) and also with upland discharge (Chatterjee 1973, Sp. Notes 1973) and are presented elsewhere. But the wind stress has been neglected in all those studies. The effect of this term is considered in this model because in the estimation of the height of embankments against flood in the river system, the incorporation of the cyclonic storms in the Bay of Bengal area, is essential.

Mathematical Formulation : Considering the propagation of the tidal wave in the river as an one dimensional problem, the equations governing the flow can be written as (Dronkers 1964).


BANDAR
Chabbis Bigha Khal
GOPIGANJ
Hurhurah Khal
Bakshi Khal
BAKSHI
KOLAGHAT
Ebb
Rupnarain River
GOPINATHPUR
JHUM JHUMI
INDEXPLAN
RUPNARAIN RIVER
Scale - 1 = 4 S. Mile
Flood
Hooghly
River
GEONKHALI

$$\frac{\partial h}{\partial x}+\frac{1}{gA}\frac{\partial Q}{\partial t}-\frac{(b+b_s)}{gA^2}\;Q\;\frac{\partial h}{\partial t}+\frac{|Q|Q}{C^2A^2R}-\frac{\tau^x}{\rho gR}=0 \qquad \ldots \quad (1)$$

and $$\frac{\partial Q}{\partial x}+b\;\frac{\partial h}{\partial t}=0 \qquad \ldots \quad (2)$$

where A=area of Cross-section, b=width and b_s=conveyance width, g=acceleration due to gravity, τ^x=Wind-stress in the x-direction, ρ=density, Q=discharge, h=water surface elevation, R=hydraulic mean depth, x=axis in the longitudinal direction, t=time, C=Friction factor.

To avoid duplication with earlier works referred above only the term T^x is discussed here.

T^x is given by $T^x=\zeta-\dfrac{V^2}{R}\cos\psi$

where ψ is the angle between the direction of the wind and the axis of x. V=wind velocity and $\zeta=\gamma^2\rho_a$ where ρ_a=density of the atmosphere and $\gamma=a$ const. Making use of the explicit finite difference method using the diamond grids (Chatterjee 1972, p. 151) and considering the rhomb $\{(2m, 2n-1), (2m-1, 2n), (2m, 2n+1), (2m+1, 2n)\}$;

eqn (1) can be written as

$$\begin{aligned}
&(h^{2n}_{2m+1}-h^{2n}_{2m-1})+\left(\frac{1}{gA}\right)^{2n}_{(2m+1,\,2m-1)}\frac{\Delta x}{\Delta t}(Q^{2n+1}_{2m}-Q^{2n-1}_{2m})\\
&-\frac{1}{4}\left(\frac{b+b_s}{gA^2}\right)^{2n}_{(2m+1,\,2m-1)}\frac{\Delta x}{\Delta t}(Q^{2n-1}_{2m}+Q^{2n+1}_{2m})\,(h^{2n}_{2m-1}-h^{2n-2}_{2m-1}+h^{2n}_{2m+1}-h^{2n-2}_{2m+1})\\
&+2\left(\frac{1}{C^2A^2R}\right)^{2n}_{(2m+1,\,2m-1)}\Delta x\,.\,|\,Q^{2n-1}_{2m}\,|\,Q^{2n+1}_{2m}\\
&-2\left(\frac{1}{gR}\right)^{2n}_{(2m+1,\,2m-1)}\Delta x\,.\,\tau^{x\,2n}_{2m}=0 \qquad \dots \quad (3)
\end{aligned}$$

and considering the rhomb {(2m+1, 2n) (2m, 2n+1), (2m+1, 2n+2), (2m+2, 2n+1)} : eqn. (2) can be written as

$$b^{2n}_{2m+1}(h^{2n+2}_{2m+1}-h^{2n}_{2m+1})+\frac{\Delta t}{\Delta x}\cdot(Q^{2n+1}_{2m+2}-Q^{2n+1}_{2m})=0 \qquad \dots \quad (4)$$

In eqns. (3) and (4) all other term excepting

$$Q^{2n+1}_{2m} \text{ and } h^{2n+2}_{2m+1}$$

are known and therefore first calculating Q^{2n+1}_{2m} using eqn. (3), its value is used to find h^{2n+2}_{2m+1} from eqn. (4).

Convergence and Stability of the finite difference scheme :

Courant *et al.* (1952) showed that for a linear system of hyperbolic eqns. the conditions for convergence as well as stability of the explicit finite difference scheme is

$$\frac{\Delta t}{\Delta x}\leqslant\frac{1}{|\,u+\sqrt{gR}\,|_{max}}\,.$$

where Δt and Δx are time and space increments.

It is still not possible to find any such condition for a non-linear set of hyperbolic eqns. and only the heuristic approach is adopted to choose Δt & Δx to make the system stable and convergent. Here Δt=120 secs and Δx=1830 m are taken.

Boundary Conditions : The water level hydrograph at the entrance of the Rupnarain (at Geonkhali) for the tidal range=6m (one of the greatest value observed here) is used as one boundary condition, while a set of different discharge conditions at Bandar the end of the Rupnarain is taken as other boundary condition. Both the unsteady and steady state discharge conditions are used to get the steady highest water levels along the river.

Initial conditions : Since the problem is studied as a mixed initial boundary value problem, the solution is mainly guided by the boundary conditions as the solution starts from the initial values being guided and modified in transit by the boundary conditions at the two ends. For this a set of approximate h and Q values is used which is rectified during iteration.

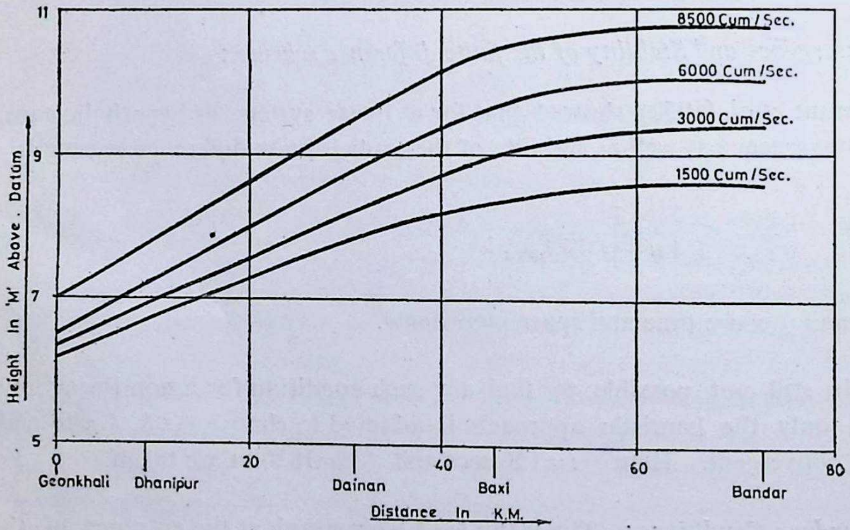
Basic Data : The river parameters viz. A, b, b_s, R and C were calculated from hydrographic survey charts, velocity and water level records in the way discussed in (Chatterjee 1974).

Wind Data : Usually wind velocity greater than 50 Km/hour is considered as above normal and from past records (available at the Regional Meteorlogical office, Alipore) it is seen that highest wind speed of the order of 90 Km/hour (approx) is recorded at Sagar (the sea face of the Hooghly estuary) and $\zeta=\gamma^2\rho_a=2.5\times10^{-6}\rho$ ($\because$ for strong wind varies from 1.5×10^{-3} to 4.0×10^{-3}) (from Miyazaki 1965) $\psi=0$ is taken to get the maximum effect of T^x.

Results : The highest water surface elevation at different points along the river are shown in fig. 2, for different upland discharges with wind velocity of the order of 90 km/hour.

Fig.2

STAGE LEVELS IN THE RUPNARAIN



Discussion : The nature of the curves in fig. 2 show that the water level from Baxi to Bandar is more or less flat while from Geonkhali to Baxi has a slope of about 1 : 13,000. From past record where the upland discharge values agree closely shows a slope of about

1 : 10,000 to 1 : 15,000. The variation depends on the wind velocity, since the model assumes the presence of both the abnormal upland discharge and abnormal wind stress simultaneously.

Conclusion : The main purpose of this model is to find out the greatest possible water level under the joint action of abnormal flood discharges and the wind stress. For this approximately the maxm wind velocity (from available data) and maxm. discharge conditions are adopted.

Acknowledgement : The author expresses his gratitude to Dr. S. K. Nag, Chief Hydraulic Engineer for his kind help and guidance.

Reference

Chatterjee, A. K. (1972) : Numerical model of a tidal river, *Bul. Cal. Math. Soc.* **64,** 151.

Chatterjee, A. K. (1972) : Mathematical Model of the Lower Reaches of Rupnarain etc., *Procd. 42nd Am. Res. Soc.* C. B. I. P., Madras.

Chatterjee, A. K. (1974) : Numerical Study of a tidal river, *Bul. Cal. Math. Soc.* **66,** 65.

Chatterjee, A. K. (1973) : Numerical Investigation of a tidal river, *Procd. Int. Symp. River Mech.*, **3,** I. A. H. R., Bangkok, 315.

Chatterjee, A. K. (1973) : Mathematical Model in the Study of Embankments of a tidal river, Sp. Note—105, Hydraulic Study Department, Calcutta Port Commissioners.

Chatterjee, A. K. (1973) : Math. Model of the Propagation of tides in a shallow Irregular Channel, *Procd. 39th Am. Conf. of Indian Math. Soc.*, Jadavpore.

Dronkers, J. J. (1964) : Tidal Computations in river and coastal area—North Holland Pub : Amsterdam.

Courant, Isacson & Rees (1952) : *Com. Pure & App. Math*, **5,** 243.

Miyazaki, M. (1965) : *The Oceanographical Magazine*, **17,** 1-2, 109.

HYDRAULIC STUDY DEPARTMENT
CALCUTTA PORT TRUST

Bull. Cal. Math. Soc., 75, 101-107 (1983)

ATTENUATION OF ACOUSTIC PLANE WAVE IN A CONDUCTING CUBIC CRYSTAL IN A MAGNETIC FIELD

N. C. Das and S. K. Bhattacharya

(*Received 9 August 1978 and Revision Received 10 March 1979*)

Abstract : The attenuation of a x-polarised y-propagating shear wave in a cubic crystal has been found out when it is acted upon by a magnetic field which acts in the direction of the propagation of wavə. The acoustic quality factor has also been determined.

1. Introduction : The propagation of magnetoelastic, magnetothermoelastic plane waves have been discussed by the researchers Kaliski [1963-a], [1963-b], Kaliski & Nowacki [1963], [1963], Paria [1967], Willson [1963], [1966], Ufliand [1963], Pospelov [1963]. Analyses published so far, however, have neglected the effects due to the passage of the wave upon the magnetic permeability of the medium. So far as the present authors are aware there is only one attempt by Wilson [1966], who has considered these effects and has shown that although the elastic strains induced cause but a small change in the numerical values of the permeability, nevertheless the effects upon wave propagation may be considerable. The aim of the present paper is a similar effort as in Wilson [1966] and it sets out to analyse the attenuation of a plane acoustic wave in an electrically conducting cubic crystal in the presence of a magnetic field. The material is assumed to be lossy in the sense that the mechanical constitutive relations have been modified by the addition of a viscous damping term.

2. Fundamental Equations : If an electrically conducting elastic solid is subjected to a mechanical load while immersed in a varying magnetic field, the electromagnetic field influences the elastic field by entering the elastic stress equation of motion as a body force called the Lorentz's pondermotive force, while the elastic field in its turn influences the electromagnetic field by modifying Ohm's law. Thus the equation of motion can be written as

$$\frac{\partial \tau_{ik}}{\partial x_k} + (\mathbf{j}\times\mathbf{B})_i = \rho \frac{\partial^2 u_i}{\partial t^2} \qquad \dots \quad (2.1)$$

The lossy constitutive equations are given by

$$\tau_{ij} = C_{ijkl}\, S_{kl} + \eta_{ijkl} \frac{\partial S_{kl}}{\partial t} \qquad \dots \quad (2.2)$$

where τ_{ik} is the mechanical strees tensor, $\mathbf{j}$ is the conduction current density vector, $\mathbf{B}$ is the magnetic induction vector, u_i are the components of the mechanical displacement vector $\mathbf{u}$, ρ

is the mass density of the medium and C_{ijkl} and η_{ijkl} are the material and viscosity constants respectively.

In the absence of displacement current and charge density, the equations governing the electromagnetic field are given by,

$$\text{Curl}\,\mathbf{H}=\mathbf{j}, \quad \text{Curl}\,\mathbf{E}=-\frac{\partial \mathbf{B}}{\partial t}$$

$$\text{div}\,\mathbf{B}=0, \quad \mathbf{j}=\sigma\left[\mathbf{E}+\left(\frac{\partial \mathbf{U}}{\partial t}\times\mathbf{B}\right)\right] \qquad \ldots \quad (2.3)$$

where **E** is the electric field vector and σ is the electrical conductivity of the medium.

For the constitutive relations between **B** and **H**, we assume that for the unstrained state of the solid.

$$\mathbf{B}=\mu_0\,\mathbf{H} \qquad \ldots \quad (2.4)$$

where μ_0 is the undisturbed scalar magnetic permeability and after displacement in the medium, we assume that the relation (2.4) takes the form (as in [7])

$$\mathbf{B}=\mu\,\mathbf{H} \qquad \ldots \quad (2.5)$$

where

$$\mu=\mu_0\,(I+\mu_0\,\delta e) \qquad \ldots \quad (2.6)$$

with the parameter δ, which is a scalar and pure number, specifying the manner in which the magnetic permeability depends upon the elastic strain, where

$$e_{ij}=\frac{1}{2}\,(u_{i,j}+u_{j,i}) \qquad \ldots \quad (2.7)$$

3. One dimensional deformation and linearisation : Let all the vectors be function of y and t only. In this case the displacement vector of an x-polarised y-propagating shear waves take the form $(u, 0, 0)$. As for the magnetic field **H**, we take this in the form

$$\mathbf{H}=\mathbf{H}_0+\mathbf{h} \qquad \ldots \quad (3.1)$$

where $\mathbf{H}_0$ is constant in space and time and **h** depends on y and t.

From (2.7), we see that the strain tensor e_{ij} takes the form,

$$e_{ij}=\frac{1}{2}\begin{pmatrix} 0 & \dfrac{\partial u}{\partial y} & 0 \\ \dfrac{\partial u}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \ldots \quad (3.2)$$

We take

$$\mathbf{H}_0=(0,\ H,\ 0) \qquad \ldots \quad (3.3)$$

and according to our problem

$$\mathbf{h}=(h_x,\ 0,\ h_z) \qquad \ldots \quad (3.4)$$

where h_x, h_z are so small that the product of h_x, h_z and u and their derivative may be neglected.

The permeability tensor is now given by

$$\mu=\mu_0 \begin{pmatrix} 1 & \frac{1}{2}\delta\frac{\partial u}{\partial y} & 0 \\ \frac{1}{2}\delta\frac{\partial u}{\partial y} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \ldots \quad (3.5)$$

Hence from equations (2.5), (2.6), (3.1), (3.3), (3.4), (3.5), we get

$$(1/\mu_0)\ \mathbf{B}=[0,\ H,\ 0]+\left[\ h_x+\frac{1}{2}\,\delta\,H\frac{\partial u}{\partial y},\ 0,\ h_z\ \right] \qquad \ldots \quad (3.6)$$

The relations (2.3) give,

$$j_x=\frac{\partial h_z}{\partial y},\quad j_y=0,\quad j_z=-\frac{\partial h_x}{\partial y} \qquad \ldots \quad (3.7)$$

while the second relation of (2.3) and (2.5) simply to

$$\frac{\partial E_x}{\partial y}=\mu_0\,\frac{\partial h_z}{\partial t}\,,\ \frac{\partial E_z}{\partial y}=-\mu_0\left(\frac{\partial h_x}{\partial t}+\frac{1}{2}\delta\ H\,\frac{\partial^2 u}{\partial y\partial t}\right) \qquad \ldots \quad (3.8)$$

Again, fourth relation of (2.3) may be written as

$$j_x=\sigma E_x,\quad j_y=\sigma\left[\ E_y-\mu_0\ h_z\frac{\partial u}{\partial t}\ \right] \qquad \ldots \quad (3.9)$$

$$j_z=\sigma\left[\ E_z+\mu_0\ H\,\frac{\partial u}{\partial t}\ \right]$$

Thus equation (3.7) and (3.9) give,

$$\sigma E_x=\frac{\partial h_z}{\partial y},\quad E_y=\mu_0\ h_z\ \frac{\partial u}{\partial t}$$

$$E_z=-\left(\ \mu_0\ H\ \ \frac{\partial u}{\partial t}+\frac{1}{\sigma}\,\frac{\partial h_x}{\partial y}\right) \qquad \ldots \quad (3.10)$$

Eliminating E_z from (3.8) and (3.10) we find

$$\frac{\partial h_x}{\partial t}=H\left(1-\frac{1}{2}\,\delta\right)\frac{\partial^2 u}{\partial y\,\partial t}+\nu_H\frac{\partial^2 h_x}{\partial y^2}$$

$$\frac{\partial h_z}{\partial t}=\nu_H\frac{\partial^2 h_z}{\partial y^2} \qquad \ldots \quad (3.11)$$

where $\nu_H=1/(\mu_0\,\sigma)$ is the magnetic viscosity.

Again from (2.2), the only non-vanishing stress τ_{xy} is given by,

$$\tau_{xy}=\left(c+\eta\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial y}$$

Thus the equation (2.1) after linearisation becomes

$$\rho\frac{\partial^2 u}{\partial t^2}=\left(c+\eta\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial y^2}+\mu_0\,H\frac{\partial^2 h_x}{\partial y^2} \qquad \ldots \quad (3.12)$$

4. Dispersion equation for finite conductivity : We assume for u, h_x and h_z

$$u=u'e^{-\alpha y}\,e^{i(\omega t-ky)}$$

$$h_x=h'_x\,e^{-\alpha y}\,e^{i(\omega t-ky)} \qquad \ldots \quad (4.1)$$

$$h_z=h'_z\,e^{-\alpha y}\,e^{i(\omega t-ky)}$$

where α, k and ω are the attenuation constant, wave number and frequency of the wave respectively.

Substituting (4.1) in (3.11) and (3.12) we get

$$i\,\omega\,H\,(\alpha+ik)\left(1-\frac{1}{2}\,\delta\right)u'-\{\nu_H(\alpha+ik)^2-i\omega\}\,h'_x=0$$

$$\{(\alpha+ik)^2\,(c+i\omega\eta)+\rho\omega^2\}u'-\mu_0\,H\,(\alpha+ik)h'_x=0 \qquad \ldots \quad (4.2)$$

$$i\,\omega\,h'_z=\nu_H\,(\alpha+ik)^2h'_z \qquad \ldots \quad (4.3)$$

We see that electromagnetic waves are from two families ; in one of them, the wave is perpendicular to the plane formed by the direction of primary magnetic field and direction of wave propagation and dispersion equation for wave of this type is

$$\omega=-i\nu_H\,(\alpha+ik)^2 \qquad \ldots \quad (4.4)$$

while wave from the other family lie in the plane formed by the direction of primary magnetic field and the direction of wave propagation ; and for them the dispersion equations is

$$\{\rho\omega^2+(c+i\omega\eta)\,(\alpha+ik)^2\}\,\{\nu_H(\alpha+ik)^2-i\omega\}-i\omega\,H^2\,\mu_0\,(\alpha+ik)^2(1-\tfrac{1}{2}\delta)=0 \qquad \ldots \quad (4.5)$$

5. Perfect Conductivity : Putting $\sigma\rightarrow\infty$ that is. $\nu_H\rightarrow 0$ in (4.5) we obtain, after separating real and imaginary parts,

$$(k^2-\alpha^2)\,\omega\eta-2\alpha k\left\{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)\right\}=0$$

$$(k^2-\alpha^2)\left\{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)\right\}+2\alpha k\omega\eta=\rho\omega^2$$

Solving for k^2 and α^2 we get

$$\alpha^2=\frac{\rho\omega^2}{2\left\{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)\right\}}\left[\frac{1}{\left\{1+\left(\frac{\omega\eta}{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)}\right)^2\right\}^{\frac{1}{2}}}-\frac{1}{\left\{1+\left(\frac{\omega\eta}{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)}\right)^2\right\}}\right] \quad \ldots \quad (5.1)$$

$$k^2=\frac{\rho\omega^2}{2\left\{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)\right\}}\left[\frac{1}{\left\{1+\left(\frac{\omega\eta}{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)}\right)^2\right\}^{\frac{1}{2}}}+\frac{1}{\left\{1+\left(\frac{\omega\eta}{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)}\right)^2\right\}}\right] \quad \ldots \quad (5.2)$$

From (5.1), we see that if *i*) $\delta<2$ the influence of the magnetic field is to attenuate the effect of the viscous coefficient. In general, for most solids, the viscosity coefficient is small and thus the intensity of the magnetic field ensures that,

$$\left[\frac{\omega\eta}{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)}\right]<<1$$

even at high frequencies. When this condition is satisfied the expressions (5.1) and (5.2) reduce to

$$\alpha^2=\frac{\rho\omega^4}{4\left\{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)\right\}}\left[\frac{\eta}{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)}\right]^2 \quad \ldots \quad (5.3)$$

$$k^2=\frac{\rho\omega^2}{\left\{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)\right\}}\left[1\mp\frac{3}{4}\left\{\frac{\omega\eta}{c+\mu_0 H^2\left(1-\frac{1}{2}\,\delta\right)}\right\}^2\right]^{-1} \quad \ldots \quad (5.4)$$

Thus the presence of the magnetic field is to introduce a second order decrease in k and, consequently, $\left(\text{since the phase velocity is given by } V=\frac{\omega}{k}\right)$ a small increase in the phase velocity. Since the wave amplitude in the lossy medium must decay in the direction of propagation, sign of α and k in (5.3) and (5.4) must both be positive for a positive travelling wave and negative for negative travelling wave. The wave number and the corresponding attenuation factor for a positive travelling wave is therefore

$$\alpha=\frac{\omega^2}{2}\left(\frac{\rho}{c+\mu_0\, H^2\left(1-\frac{1}{2}\,\delta\right)}\right)^{\frac{1}{2}}\left(\frac{\eta}{c+\mu_0\, H^2\left(1-\frac{1}{2}\,\delta\right)}\right) \quad \ldots \quad (5.5)$$

$$K=\omega\left(\frac{\rho}{c+\mu_0\, H^2\left(1-\frac{1}{2}\,\delta\right)}\right)^{\frac{1}{2}}\left(1+\frac{3}{4}\left\{\frac{\omega\eta}{c+\mu_0\, H^2\left(1-\frac{1}{2}\,\delta\right)}\right\}^2\right)^{-\frac{1}{2}} \quad \ldots \quad (5.6)$$

The attenuation per wave length

$$\alpha\lambda=\frac{2\pi\alpha}{\lambda} \quad (\lambda=\text{wave length})$$

is a function of the parameter

$$Q=\frac{c+\mu_0\, H^2\left(1-\frac{1}{2}\,\delta\right)}{\omega\eta}$$

which is called the acoustic quality factor. Obviously the acoustic quality factor of the solid increases when it is influenced by the magnetic field.

ii) For values of δ gradually greater than 2, the values of the attenuation constant and the wave number will also gradually increase.

iii) It is interesting to note that for $\delta=2$, there will be no influence of the magnetic field on α and k. This case will correspond to the classical solution obtained by Auld [1973].

We do not have available detailed knowledge of the values of δ, which indeed may well depend upon frequency. It follows, therefore, that the effect of the strain on magnetic permeability may have important consequences.

References

Auld, B. A. (1973) : Acoustic fields and waves in solids, 1, John Willy & Sons.

Kaliski, S. (1963-a) : *Arch. Mech. Stos.* **15**, 507.

Kaliski, S. (1963-b) : *Proc. Vibr. Probl.* **4**(1), 85.

Kaliski, S. & Nowacki, W. (1962) : *Bull. Acad. Poln. Ser. Tech.* **10**, 25.

Kaliski, S. & Nowacki, W. (1963) : *Int. J. Eng. Sc.* **1**, 163.

Paria, G. (1967) : Advances in Applied Mechanics, 10, Academic Press.

Pospelov, L. A. (1963) : *Prink. Mat. Mech.* **27**, 939, translated as PMM. *J. Appl. Math. Mech.* **27**, 1437 (1964).

Ufliand, Y. S. (1963) : *Prink. Mat. Mech.* **27**, 740, translated as PMM. *J. Appl. Math. Mech.* **27**, 1135 (1964).

Willson, A. J. (1963) : *Proc. Camb. Phil. Soc.* **59**, 483.

Willson, A. J. (1966) : *Proc. Camb. Phil. Soc.* **62**, 91.

DEPARTMENT OF MATHEMATICS
JADAVPUR UNIVERSITY
CALCUTTA-32, INDIA.

Bull. Cal. Math. Soc., **75**, 109-115 (1983)

SMALL RADIAL VIBRATION OF FINITELY DEFORMED LONG ELASTIC CYLINDER

SRIPATI RANJAN CHAUDHURI

(*Received 27 May 1978 and 15 December 1978*)

Abstract : Small radial vibration superposed on finitely deformed (radially) long circular isotropic elastic cylinder is studied and ultimately a condition is obtained for existence of vibration. The condition so obtained is satisfied with some numerical values and hence it shows that finitely deformed state may be stable with respect to radial disturbance.

1. **Introduction.** Very few problems concerning small amplitude vibrations of initially stressed bodies have been studied. In 1961 Green (1961) first considered a problem of small torsional vibration over finite extension of a circular cylinder. Later in 1965 Suhubi (1965) considered small longitudinal vibration of an initially stretched circular cylinder. In 1970 a joint paper of Demiary and Suhubi (1970) was on small torsional vibration of an initially twisted circular rubber cylinder. At present an attempt is made to observe what happens if small radial vibration is superposed on radially finitely-deformed long isotropic homogeneous incompressible elastic circular cylindrical shell. It should be noted that at present, differing from past papers, finite extension ratio is not constant. For finite deformation of the cylindrical shell force is applied on the outer surface, inner surface being free. No extra force is used to cause small radial vibration over the finite radial displacement. It may be called a kind of free small oscillation superposed on large deformation. Ultimately governing equations are simplified by considering rubber like Mooney-material and frequency equation is obtained. From frequency equation we arrive at a condition for small vibration in reality. This obtained condition is found to be valid in some numerical cases. Notations and theories of Green and Zerna (1961) are used to obtain equations.

2. **Finitely Deformed State :** Let us consider a long isotropic homogeneous circular cylinder having inner and outer radii r_1 and r_2 in the undeformed state (body B^0). We assume that the cylinder suffers a finite radial displacement and in the deformed state (body B) R_1 and R_2 be respective values of r_1 and r_2. Using cylindrical coordinates, Z-axis being the axis of the cylinder a typical point (r,θ,z) assumes the point (R,θ,z) and we take

$$Q(R)= \frac{r}{R} \qquad \dots \quad (2.1)$$

This radial deformation is obtained by the application of a pressure q uniformly distributed over the outer surface of the cylinder.

Incompressibility condition will be

$$R^2-R_1^2=r^2-r_1^2 \qquad \dots \quad (2.2)$$

From (2.1) and (2.2) we get

$$\frac{dQ}{dR}=\frac{1}{R}\,\frac{1-Q^2}{Q} \qquad \dots \quad (2.3)$$

Now metric tensors G_{ij}, G^{ij}, g_{ij}, g^{ij}, and strain invariants I_1, I_2, I_3 can be obtained. Utilizing these in the constitutive eqns. (for incompressible material)

$$T^{ij}=\Phi\, g^{ij}+\Psi\, B^{ij}+p\, G^{ij} \qquad \dots \quad (2.4)$$

where

$$B^{ij}=I_1 g^{ij}-g^{ir}\, g^{js}\, G_{rs}$$

$$\Phi=2\,\frac{\partial W}{\partial I_1},\ \Psi=2\,\frac{\partial W}{\partial I_2}$$

W= strain energy function $=W(I_1, I_2)$, p=hydrostatic pressure

$$I_1=g^{rs}G_{rs},\ I_2=g_{rs}G^{rs}I_3,\ I_3=\frac{G}{g}=1$$

stresses can be obtained. Equations of equilibrium in cylindrical coordinates and boundary conditions

$$T^{11}=0 \text{ on } R=R_1 \text{ and } T^{11}=-q \text{ on } R=R_2 \qquad \dots \quad (2.5)$$

will lead to the solution as obtained by particularizing eqns. in Art. (3.4) of [Green and Zerna 1961].

$$\left.\begin{aligned} &T^{11}=\Phi Q^2+\Psi(1+Q^2)+p,\ T^{22}=\Phi/Q^2R^2+\Psi(1+Q^2)/Q^2R^2+p/R^2 \\ &T^{33}=\Phi+\Psi(1+Q^4)/Q^2+p,\ T^{12}=T^{23}=T^{31}=0 \end{aligned}\right\} \qquad \dots \quad (2.6)$$

$$p=-Q^2\Phi-(1+Q^2)\,\Psi-L(Q)-q \qquad \dots \quad (2.7)$$

$$L(Q)=\int_{Q}^{Q_2} \frac{1+Q^2}{Q}\,(\Phi+\Psi)\,dQ \qquad \dots \quad (2.8)$$

$$L(Q_1)=-q \qquad \dots \quad (2.9)$$

$$Q_1=\frac{r_1}{R_1},\ Q_2=\frac{r_2}{R_2} \qquad \dots \quad (2.10)$$

$$I_1 = I_2 = 1 + Q^2 + \frac{1}{Q^2} \qquad \dots \quad (2.11)$$

For a given force q, the relation (2.9) and incompressibility condition

$$R_2^2 - R_1^2 = r_2^2 - r_1^2 \qquad \dots \quad (2.12)$$

will determine the radii R_1 and R_2 after deformation.

3. **Small Superposed Radial Vibration State.** Let us assume now that the finitely deformed cylinder assumes a small additional deformation $\epsilon \underset{\sim}{\omega}(R,\theta,z,t)$, ϵ is small quantity so that ϵ^2 and higher powers can be neglected. Let the body has gone to the state B'. We are interested in superposed small radial vibration and so we take contravariant components of $\underset{\sim}{\omega}$ as

$$\omega^1 = f(R)\, e^{ikt}, \quad \omega^2 = \omega^3 = 0 \qquad \dots \quad (3.1)$$

$$\therefore \quad \omega_1 = \omega^1 = f(R)\, e^{ikt}, \quad \omega_2 = R^2 \omega^2 = 0, \quad \omega_3 = \omega^3 = 0$$

Now all the quantities G_{ij}, G^{ij}, B^{ij}, T^{ij}, I_1, I_2, I_3, p in the finite deformed state will change to

$$G_{ij} + \epsilon\, G'_{ij}, \quad G^{ij} + \epsilon\, G'^{ij}, \quad T^{ij} + \epsilon\, T'^{ij}$$

$$I_1 + \epsilon\, I'_1, \quad I_2 + \epsilon\, I'_2, \quad I_3 + \epsilon\, I'_3, \quad p + \epsilon\, p'$$

Increments can be calculated by the formulae of Art. 4.1 of [Green and Zerna 1961]. It is found that

$$I'_3 = I_3\, G^{ij} G'_{ij} = 2\left(\frac{df}{dR} + \frac{f}{R}\right) e^{ikt}$$

Therefore incompressibility condition $I'_3 = 0$ will give

$$f = A/R\,, \ A \text{ is a constant} \qquad \dots \quad (3.2)$$

Utilizing (3.2)

$$G'_{ij} = \text{diag}\left[-\frac{2A}{R^2}\, e^{ikt},\ 2Ae^{ikt},\ 0\right] \quad G'^{ij} = \text{diag}\left[\frac{2A}{R^2}\, e^{ikt},\ -\frac{2A}{R^4}\, e^{ikt},\ 0\right] \qquad \dots \quad (3.3)$$

$$B'^{ij} = \text{diag}\left[\frac{2A}{R^2}\, e^{ikt},\ -\frac{2A}{R^4}\, e^{ikt},\ \frac{2A}{R^2}\left(\frac{1-Q^4}{Q^2}\right) e^{ikt}\right] \qquad \dots \quad (3.4)$$

and increments of stresses referred to the state B as

$$\left.\begin{aligned}
\lambda^{11} &= Q^2\Phi' + (Q^2+1)\Psi' + Ae^{ikt}\left(-\frac{Q^2}{R^2}\Phi + \frac{1-Q^2}{R^2}\Psi + \frac{p}{R^2}\right) + p' \\
\lambda^{22} &= \frac{\Phi'}{Q^2R^2} + \frac{(Q^2+1)}{Q^2R^2}\Psi' + Ae^{ikt}\left(\frac{1-Q^2}{Q^2R^4}\Psi + \frac{\Phi}{Q^2R^4} - \frac{p}{R^4}\right) + \frac{p'}{R^2} \\
\lambda^{33} &= \Phi' + \frac{Q^4+1}{Q^2}\Psi' + 2Ae^{ikt}\,\frac{(1-Q^4)}{R^2Q^2}\,\Psi\, p'
\end{aligned}\right\} \qquad \dots \quad (3.5)$$

$$\lambda^{ij}=0,\ i\neq j$$

where

$$\lambda^{ij}=T'^{ij}+T^{im}\omega^{j}\,||_{m}+T^{ij}\omega^{m}\,||_{m}\ (\ ||\ \text{stands for covariant diff.})$$

$$\left.\begin{aligned}&T'^{ij}=g^{ij}\,\Phi'+B^{ij}\,\Psi'+B'^{ij}\,\Psi+G'^{ij}p+G^{ij}p'\\&\Phi'=A_1I'_1+F_1I'_2,\ \Psi'=F_1I'_1+B_1I'_2\\&A_1=2\,\frac{\partial^2W}{\partial I_1^2},\ B_1=2\,\frac{\partial^2W}{\partial I_2^2}\ ,\ F_1=2\,\frac{\partial^2W}{\partial I_1\partial I_2}\end{aligned}\right\}\quad\cdots\quad(3.6)$$

Equations of motion without body forces [eqn. 4.1.18 of [Green and Zerna 1961]] will become

$$\left.\begin{aligned}&\frac{\partial\lambda^{11}}{\partial R}+\frac{\lambda^{11}-R^2\lambda^{22}}{R}=-\frac{\rho AK^2}{R}\ e^{ikt}\\&\frac{\partial\lambda^{22}}{\partial\theta}=0\ ,\ \frac{\partial\lambda^{33}}{\partial z}=0\end{aligned}\right\}\quad\cdots\quad(3.7)$$

Last two equations of (3.7) show that p' will be independent of θ & z. Boundary conditions [relation 4.1.23 of [Green and Zerna 1961]] will be

$$\lambda^{11}=0\ \text{on}\ R=R_1\ \&\ R=R_2\quad\cdots\quad(3.8)$$

since $T^{11}=0$ on $R=R_1$ and $T^{11}=-q$ on $R=R_2$ are already satisfied in the finite deformed state.

4. **Simplification for Mooney Material.** For Mooney-material strain-energy function is given by

$$W=C_1(I_1-3)+C_2(I_2-3),\ C_1,\ C_2\ (\geqslant 0)\ \text{constants}\quad\cdots\quad(4.1)$$

and so

$$\Phi=2C_1,\ \Psi=2C_2,\ \Phi'=\Psi'=0\quad\cdots\quad(4.2)$$

Using (3.2) results in the seotions (3) and (4) will become

$$\left.\begin{aligned}&T^{11}=2C_1Q^2+2C_2(1+Q^2)+p,\ T^{22}=2C_1/Q^2R^2+2C_2\,\frac{1+Q^2}{Q^2R^2}+\frac{p}{R^2}\\&T^{33}=2C_1+2C_2\,\frac{Q^4+1}{Q^2}+p,\ T^{12}=T^{23}=T^{31}=0\end{aligned}\right\}\quad\cdots\quad(4.3)$$

$$q=2\,(C_1+C_2)\left[\ \log\frac{Q_2}{Q_1}+\frac{Q_1^2-Q_2^2}{2}\ \right]\quad\cdots\quad(4.4)$$

where $Q_1=r_1/R_1$ and $Q_2=r_2/R_2$

$$p=C_1[2\log Q-Q^2-2\log Q_1-Q_1^2]+C_2[2\log Q-Q^2-2\log Q_1-Q_1^2-2] \quad \ldots \quad (4.5)$$

$$\frac{dp}{dR}=2(C_1+C_2)\,\frac{(1-Q^2)^2}{Q^2R} \qquad \text{using (2.3)} \quad \ldots \quad (4.6)$$

$$\lambda^{11}=Ae^{ikt}\left(-\frac{2Q^2}{R^3}\,C_1+\frac{1-Q^2}{R^2}\,2C_2+\frac{p}{R^2}\right)+p' \quad \ldots \quad (4.7)$$

$$\left.\begin{aligned}\lambda^{22}&=Ae^{ikt}\left[\frac{2C_1}{R^4Q^2}+\frac{2C_2}{R^4}\,\frac{1-Q^2}{Q^2}-\frac{p}{R^4}\right]+\frac{p'}{R^2}\\ \lambda^{33}&=\frac{4AC_2}{R^2}\,\frac{(1-Q^4)}{Q^2}\,e^{ikt}+p'\end{aligned}\right\} \quad \ldots \quad (4.8)$$

$$\frac{\lambda^{11}-R^2\lambda^{22}}{R}=\frac{2Ae^{ikt}}{R}\left[-\frac{C_1}{R^2}\,\frac{Q^4+1}{Q^2}-\frac{C_2(1-Q^2)^2}{R^2Q^2}+\frac{p}{R^2}\right] \quad \ldots \quad (4.9)$$

Using (4.5), (4.6) & (2.3) 1st. relation of (4.7) after diff. w.r.t. R will give on simplification

$$\frac{d\lambda^{11}}{dR}=\frac{2A(C_1+C_2)e^{ikt}}{R^3}\left[Q^{-2}(6Q^4-4Q^2+1)-2\log Q+2\log Q_1+Q_1^2\right]+\frac{dp'}{dR} \quad .. \quad (4.10)$$

5. **Solution of the Problem.** Putting the value of p from (4.5) in (4.8) eqn. of motion (3.7) becomes

$$\begin{aligned}&\frac{2A(C_1+C_2)e^{ikt}}{R^3}\left[Q^{-2}(6Q^4-4Q^2+1)-2\log Q+2\log Q_1+Q_1^2\right]\\ &+\frac{dp'}{dR}+\frac{e^{ikt}}{R^3}\Big[-2AC_1Q^{-2}(Q^4+1)-2AC_2Q^{-2}(1-Q^2)^2\\ &+2A\{C_1(2\log Q-Q^2-2\log Q_1-Q_1^2)+C_2(2\log Q-Q^2-2-Q_1^2-2\log Q_1)\}\Big]\\ &\qquad\qquad=-\rho\,\frac{K^2A}{R}\,e^{ikt}\end{aligned}$$

which on simplification gives

$$\frac{8A(C_1+C_2)}{R^3}\,(Q^2-1)+\frac{\rho K^2A}{R}+\frac{dp'}{dR}\,e^{-ikt}=0 \quad \ldots \quad (5.1)$$

Boundary condition (3.8) become

$$\frac{A(C_1+C_2)e^{ikt}}{R_2^2}\left[2\log Q_2-3Q_2^2-2\log Q_1-Q_1^2\right]+(p')_{R=R_2}=0 \quad \ldots \quad (5.2)$$

and

$$\frac{A(C_1+C_2)}{R_1^2}\,e^{ikt}\,[-4Q_1^2]+(p')_{R=R_1}=0 \quad \ldots \quad (5.3)$$

Integrating the eqn. (5.1) from $R=R_1$ to $R=R_2$ we get

$$8A(C_1+C_2)\int_{R_1}^{R_2}\frac{Q^2-1}{R^3}\,dR+\rho K^2A\Big[\log R\Big]_{R_1}^{R_2}+e^{-ikt}\Big[(p')_{R=R_2}-(p')_{R=R_1}\Big]=0 \quad \dots (5.4)$$

From incompressibility condition (2.2)

$$Q^2=\frac{r^2}{R^2}=\left(1-\frac{R_1^2-r_1^2}{R^2}\right)$$

and therefore

$$\int_{R_1}^{R_2}\frac{Q^2-1}{R^3}\,dR=(r_1^2-R_1^2)(R_2^4-R_1^4)/4R_1^4R_2^4 \quad \dots (5.5)$$

Substituting (5.5) and the values of $(p')_{R=R_2}$, $(p')_{R=R_1}$ from (5.2) & (5.3) to the eqn. (5.4) we get the equation as

$$\frac{8(C_1+C_2)(r_1^2-R_1^2)(R_2^4-R_1^4)}{4R_1^4R_2^4}+\rho K^2\log\frac{R_2}{R_1}=\frac{C_1+C_2}{R_2^2}\left[2\log\frac{r_2R_1}{r_1R_2}-\frac{3r_2^2}{R_2^2}-\frac{r_1^2}{R_1^2}\right]+(C_1+C_2)\frac{4r_1^2}{R_1^4}$$

where we have used the values

$$Q_2=\frac{r_2}{R_2}\,,\ Q_1=\frac{r_1}{R_1}\,.$$

On simplification we get

$$\rho K^2\log\frac{R_2}{R_1}=\frac{2(C_1+C_2)}{R_2^2}\log\frac{r_2R_1}{r_1R_2}+\frac{C_1+C_2}{R_1^4R_2^4}\Big[2R_1^4(r_1^2-R_1^2)+r_1^2R_2^2(R_2^2-R_1^2)+r_1^2R_2^4-3R_1^4r_2^2+2R_1^2R_2^4\Big] \quad \dots (5.6)$$

This is the frequency equation. It will give two values of K. Using the value of K in (5.1) and then integrating we can have the expression for p'.

In (5.6) it is difficult to show that K^2 always assumes positive values. Therefore to have real existence of periodic oscillation the condition is that K^2 should be positive i.e.

$$2R_1^4R_2^2\log\frac{r_2R_1}{r_1R_2}+2\Big[2R_1^4(r_1^2-R_1^2)+r_1^2R_2^2(R_2^2-R_1^2)+r_1^2R_2^4-3R_1^4r_2^2+2R_1^2R_2^4\Big]>0 \quad \dots (5.7)$$

since C_1 & $C_2 \geqslant 0$ & $\log \dfrac{R_2}{R_1} > 0$. R_2 & R_1 are obtained from (2.12) & (4.4)

Considering simple numerical cases it is observed that the condition (5.7) is valid and corresponding periods are obtained. Numerical values are given in the following table

r_1	r_2	q/C_1+C_2	R_1	R_2	$\rho K^2/C_1+C_2$
1.5	2	.99	1	1.66	19.84
1.5	2.5	.43	1.25	2.36	2.93
2	4	.72	1.5	3.77	.71
4	7	.12	3	6.48	.20

$R_2 < r_2$ and hence $R_1 < r_1$ since the outer surface is acted on by pressure.

6. **Conclusion.** In the conclusion it is remarked that a long circular incompressible isotropic homogeneous elastic cylinder after having been deformed finitely in the radial direction by an external pressure on the outer surface may execute a further free small radial oscillation. This shows that the equilibrium in the finite deformed state is stable with respect to radial disturbance.

Acknowledgement. Author remains grateful to his teacher Dr. S. K. Chakraborty of Burdwan University for his suggestion and guidance.

References

Demiary H. and Suhubi E. S. (1970) : *Int. J. Eng. Sci.*

Green A. E and Zerna W. (1961) : Theoretical elasticity, Oxford.

Green A. E. (1961) : Contributions in honor of the 70th. birth day of N. I. Muskheslishvili.

Suhubi E. S. (1965) : *Int. J. Eng. Sc.* **2**, 509.

B. N. COLLEGE, ITACHUNA
HOOGHLY, WEST BENGAL.

Bull Cal. Math. Soc., 75, 117-127 (1983)

THE STABILITY OF VERTICALLY STRATIFIED NON-CONDUCTING ISENTROPIC SHEAR FLOW BETWEEN TWO PARALLEL PLATES

B. M. Rastogi and S. C. Agrawal

(*Received 20 October 1978, Revision Received 12 September 1979*)

1. Introduction : The stability of parallel flows between two plates has been studied extensively by many authors. Because of its practical importance, the problem is further generalized so as to include the effects of various important physical quantities such as magnetic field, temperature gradient etc. In fact, the flow of water in a river or a canal serves as a very good example of this type of flow. Not only this, the study of atmosphere and atmospheric stability is being made successfully by constructing a model of this type.

There are various other physical phenomena which need further investigation. One such physical phenomenon which we feel should be studied in this context is the entropy of fluid. In order that the clear cut effect of entropy gradient is obtained, we have tried to simplify the model by neglecting some other effects. Thus we have assumed the fluid to be non-viscous and thermally non-conducting. The effects of compressibility are retained, so that our model can serve to some extent as a good approximation to some atmospheric studies.

The origin of the problem perhaps dates back to Lord Rayleigh (1900) and Lord Kelvin (1910) who posed the problem in its simplest form.

Recently, Miles (1957, 1959, 1961), Howard (1961), Case (1960), Dyson (1960) and Drazin (1958) etc. have contributed to the hydrodynamic and hydromagnetic stability of heterogeneous shear flows in which a non-viscous, homogeneous or non-homogeneous fluid flows between two parallel plates of infinite lengths. A majority of them have ignored the effects of gravity, which on the contrary, have an important bearing on the stability of the system and often give a deep insight into the problem. The problem was first of all beautifully formulated by Miles (1961) in terms of non-dimensional parameter J, called the Richardson number, and sufficient conditions for the existence of stable modes were obtained. In a series of papers already referred, they made the critical study of complex wave velocity, its bounds, and the existence of stable, unstable and neutral modes. Howards semi-circle-theorem provides the bounds of the complex wave velocity of an arbitrary unstable mode. However, the effects of compressibility of the fluid, on the other hand, are still uninvestigated in most of the cases whereas all real fluids are compressible to a varying degree. The change of state, however, takes place either isothermally or adiabatically.

For compressible stratified parallel flow, Warren (1968) obtained a sufficient condition for stability, but his expression is weaker than the expected generalization of the Miles sufficient condition and, as he indicated, is not the optimum condition. Further, Chimonas (1970) showed how the incompressible fluid results could be extended to gravitationally stratified compressible fluid containing a parallel flow. The importance of the Chimonas condition lies in the fact that this is a very natural generalization of Miles Howard theorem. Some contribution has also been made in this direction by Eckart (1963), Blumen (1970), Blumen, Drazin Billings (1975) and Drazin and Davey (1977). For different geometries, the problem has been investigated by Saini (1974), Howard (1973) and Lalas (1975).

The above investigations are of great physical significance in as much as the efforts are made to understand physically more realistic situations. It will therefore, be a significant contribution if such efforts are made extensively in various problems of physical importance. With these motivations, therefore, the stability of a plane, heterogeneous, shear flow of compressible and non-dissipative fluid is investigated in the present paper. The motion has been taken to be adiabatic.

In Part *A*, the problem has been investigated in its full generality. Parts-*B*, *C* and *D* respectively investigates the stability for subsonic disturbances, supersonic disturbances and fluid at rest. Sufficient conditions and other significant results have been obtained in the above mentioned situations. Results obtained in the present paper demonstrate a clear-cut effect of entropy gradient.

Formulation : For an adiabatic, non-viscous compressible flow, the fundamental equations to be used are

$$\rho \frac{D\underline{u}}{Dt} = -\nabla p - g\,\underline{\lambda}\,\rho \qquad \ldots \quad (2.1)$$

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0, \qquad \ldots \quad (2.2)$$

$$\frac{DS}{Dt} = 0, \qquad \ldots \quad (2.3)$$

$$\gamma \log \rho = \log p - S/C_v. \qquad \ldots \quad (2.4)$$

Here g is the gravitational constant and $\underline{\lambda} = (0, 0, 1)$. The quantities $\underline{u}$, ρ, p, and S are respectively the velocity, density, pressure and entropy. $\gamma = c_p/c_v$ is the ratio of specific heats. These equations are linearized about a mean velocity $[U(z), 0, 0]$, and a mean density, pressure and entropy $\rho_0(z)$, $p_0(z)$ and $S_0(z)$ by the equations

$$\underline{u} = (U+u', v', w'),\ \rho = \rho_0+\rho'$$

$$p = p_0+p',\quad S = S_0+S'_0.$$

where the primed quantities, called the perturbations, are assumed to be small and the basic quantities $\rho_0(z)$, $p_0(z)$ and $S_0(z)$ satisfy the equations

$$\frac{\partial p_0}{\partial z} + g\rho_0 = 0,$$

and

$$\gamma \log \rho_0 = \log p_0 - S_0/C_v.$$

For any perturbation quantity $f(x, y, z, t)$, we assume the form

$$f(x, y, z, t) = f(z) \exp [i(\alpha x+\beta y - \alpha ct),$$

substitute in the linearized perturbation equations and eliminate various quantities. We get the following pair of non-dimensional equations.

$$\left(\frac{\rho w}{U-c}\right)' + \frac{{}^{M}S_0}{U-c}\, w = \frac{i\,\alpha}{(U-c)^2}\left\{\frac{k^2}{\alpha^2} - \frac{(U-c)^2}{V_S^2}\right\} p \ \ldots \qquad \ldots \quad (2.5)$$

and

$$\frac{i\,\rho_0}{\alpha(U-c)}\,\{F_1 - \alpha^2\,(U-c)^2\}\, w = Dp+Mp \ \ldots \qquad \ldots \quad (2.6)$$

where

$$F_1 = \frac{dg\ DS_0}{\gamma C_v\ U^{*2}}\,,\ M_{S_0} = \frac{DS_0}{\gamma C_v} \text{ and } M = \frac{dg}{V_s^2}$$

are non-dimensional parameters. $k^2 = \alpha^2+\beta^2$ is the total wave number. $V_s^2 = \gamma p_0/\rho_0$ is the square of local sonic velocity and U^* is the characteristic velocity which can be taken as the average of the undisturbed flow. These equations are to be solved with the help of boundary conditions

$$w = 0 \text{ at } z = \pm 1.$$

We now introduce the following transformations

$$p = P \exp \left[- \int Mdz \right] \text{ and } G = \rho_0\, w\, (U-c) \exp \left[-M_{S_0}\, dz \right].$$

Equations (2.5) and (2.6) are then reduced to

$$DG= \frac{i\,\alpha Q}{(U-c)^2}\left\{\frac{k^2}{\alpha^2} - \frac{(U-c)^2}{V_s^2}\right\} P, \qquad \ldots \quad (2.7)$$

$$DP= \frac{i}{\alpha\, Q}\,\{ F_1 - \alpha^2\, (U-c)^2 \}\, G, \qquad \ldots \quad (2.8)$$

where

$$Q = \exp \int (M_{s_0} - M)\, dz.$$

The boundary conditions are

$$G=0 \text{ at } z=\pm 1.$$

3. *PART-A General Theory.*

THEOREM 3.1 : *The non-oscillatory modes, if exist, are stable.*

Proof. To prove this, we eliminate P from equations (2.7) and (2.8) to get one equation in G, namely,

$$D\left\{\frac{DG}{i\alpha Q\left(\frac{k^2}{\alpha^2(U-c)^2}-\frac{1}{V_s^2}\right)}\right\}=\frac{i}{\alpha Q}\{F_1-\alpha^2(U-c)^2\}G \qquad \ldots \quad (3.1)$$

Multiply this equation by G^*, the complex conjugate of G, integrate within the limits of z and separate the imaginary part of the resulting equation. We get

$$c_i \int_{-1}^{1} \left\{\frac{k^2\, \alpha^2\, V_s^2}{\mathcal{Q}\, A}\, |DG|^2+\frac{\alpha^2}{\mathcal{Q}}\, |G|^2\right\}(U-c_i)dz=0 \qquad \ldots \quad (3.2)$$

This shows that if $c_r=0$ then $c_i=0$. In other words, non-oscillatory modes if exist, shall be stable.

THEOREM 3.2 : *If the entropy gradient DS_0 is every where negative and $M_v > 1$ for all z, then the modes are oscillatory.*

Proof : Multiply equation (2.7) by P^*, the complex conjugate of P, integrate over the range of z, and substitute the value of DP^* from equation (2.8). This yields

$$\int_{-1}^{1} \left(\frac{F_1 \mid G \mid^2}{\mathcal{Q}}+\frac{\alpha\mathcal{Q} \mid P \mid^2}{V_s^2}\right) dz= \int_{-1}^{1} \left(\frac{\mathcal{Q}\, k^2 \mid P \mid^2}{\mid U-c \mid^2}+\frac{\alpha \mid G \mid^2}{\mathcal{Q}}\right) (U^2-2Uc^*+c^{*2})dz \qquad \ldots \quad (3.3)$$

The real part of this equation gives

$$\int_{-1}^{1} \left\{\frac{F_1 \mid G \mid^2}{\alpha\, \mathcal{Q}}+\frac{\mathcal{Q} \mid P \mid^2}{V_s^2}\right\}dz$$

$$=\int_{-1}^{1} \left\{\frac{\mathcal{Q}\, k^2 \mid P \mid^2}{\alpha \mid U-c \mid^4}+\frac{\alpha \mid G \mid^2}{\mathcal{Q}}\right\} (U^2-2c_r\, U+c_r^2-c_i^2)\, dz \qquad \ldots \quad (3.4)$$

Now let the modes be stable and non-oscillatory so that $c_r=0$ and $c_i=0$. Equation (3.4) then reduces to

$$\int_{-1}^{1} \frac{k^2}{\alpha V_s^2}\left(M_v - \frac{\alpha^2}{k^2}\right) | P |^2 \, dz = \int_{-1}^{1} \left(\frac{F_1}{\alpha \mathfrak{Q}} - \frac{\alpha U^2}{\mathfrak{Q}}\right) | G |^2 \, dz \qquad \dots \quad (3.5)$$

where $M_v = V_s^2 / U^2$.

Equation (3.5) is mathematically inconsistent if the conditions

$$DS_0 < 0 \text{ and } M_v > 1 \qquad \dots \quad (3.6)$$

hold every where in the flow domain, showing, thereby that both c_r and c_i can not be zero simultaneously. To be specific, $c_r=0 \Rightarrow c_i \neq 0$. Therefore, the non-oscillatory modes, if exist, are unstable under these condition. But theorem 3.1 shows that the non-oscillatory modes are stable irrespective of any condition on DS_0 and M_v. Obviously, this unhappy and contradictory situation arises because of our assumption about the existence of non-oscillatory modes. We conclude that the non-oscillatory modes can not exist under the conditions (3.6). Thus the modes are oscillatory under (3.6).

THEOREM 3.3 : *If DS_0 be everywhere positive and $M_v > 1$ for all z then the modes are either oscillatory or non-oscillatory unstable for the wave numbers given by $\alpha^2 < F_1/U^2$.*

Proof : Proof follows as in theorem 3.2.

THEOREM 3.4 : *The complex wave velocity of an arbitrary unstable modes, if exists, for positive entropy gradient DS_0 lies within a semicircle in the $c_r - c_i$ plane with centre at origin and radius being equal to the maximum of the basic flow U(z).*

Proof : We multiply equation (3.3) by c and equate to zero the imaginary part of the resulting equation. This yields

$$c_i \left[\int_{-1}^{1} \left\{\frac{F_1 | G |^2}{\alpha Q} + \frac{\alpha \mathfrak{Q} | P |^2}{V_s^2}\right\} dz \right.$$
$$\left. - \int_{-1}^{1} \left\{\frac{Q k^2 | P |^2}{\alpha | U - c |^2} + \frac{\alpha | G |^2}{Q} \quad (U^2 - | c |^2)\right\}\right] = 0. \qquad \dots \quad (3.7)$$

Let under the conditions stated in the theorem, the modes be unstable so that $c_i \neq 0$. From equation (3.7), we must necessarily have

$$\int_{-1}^{1} \left\{\frac{F_1 | G |^2}{\alpha Q} + \frac{\alpha Q | P |^2}{V_s^2}\right\} dz$$
$$+ \int_{-1}^{1} \left\{\frac{Q k^2 | P |^2}{\alpha | U - c |^4} + \frac{\alpha | G |^2}{Q}\right\} (| c |^2 - U^2) \, dz = 0. \qquad \dots \quad (3.8)$$

In order that the equation (3.8) is mathematically consistent, we must necessarily have

$$| c |^2 - U^2 < 0 \text{ some where in the flow domain,}$$

or
$$c_r^2 + c_r^2 < U_{s_0}^2 \leqslant U_{max}^2,$$

where z_0 is some point in the flow domain. This proves the theorem.

4. **Part-B Analysis for Subsonic Disturbances :** As defined by Lesson, Fox and Zein (1965), a disturbance is defined as subsonic or supersonic according as the relative phase velocity $U-c$ is less than or greater than the sound speed ($U-c <> V_s$), Therefore, for subsonic disturbances, equation (2.7) reduces to

$$\alpha^2 (U-c)^2 \, DG = i\alpha \, \mathfrak{Q} \, k^2 \, P \qquad \dots \quad (4.1)$$

Equation (2.8) remains unchanged and on eliminating P from equation (2.8) and (4.1) we get

$$D\left\{ \frac{(U-c)^2}{\mathfrak{Q}} \, DG \right\} = \left\{ \frac{k^2}{\alpha^2 \, \mathfrak{Q}} F_1 - \alpha^2 (U-c)^2 \right\} G \qquad \dots \quad (4.2)$$

Further the transformation $G = (U-c)^{-1/2} \phi$

when applied to equation (4.2) yields

$$\left[\frac{(U-c)\,\phi'}{Q} \right] - \frac{U'^2 \phi}{4\,Q(U-c)} - \frac{U'' \phi}{2Q} + \frac{1}{2} \frac{U'}{Q} \left(M_{s_0} - \frac{gd}{V_s^2} \right)$$
$$- \frac{K^2}{\alpha^2 \, Q} \left(\frac{F_1}{U-c} - \alpha^2 (U-c) \right) \phi = 0 \qquad \dots \quad (4.3)$$

The boundary conditions are

$$\phi = 0 \text{ at } z = \pm 1.$$

THEOREM 4.1 : *If the entropy gradient DS_0 is everywhere negative, the system is stable.*

Proof: Multiply equation (4.3) by ϕ^*, the complex conjugate of ϕ, integrate over the range of z and separate the imaginary part of the resulting equation. This gives

$$c_i \left\{ \int_{-1}^{1} \frac{| \phi' |^2 + k^2 | \phi |^2}{\mathfrak{Q}} \, dz + \int_{-1}^{1} \frac{1}{\mathfrak{Q} \, | U-c |^2} \left(\frac{U'^2}{4} - \frac{k^2 F_1}{\alpha^2} \right) | \phi |^2 \, dz \right\} = 0$$
$$\dots \quad (4.4)$$

It follows that if DS_0 is everywhere negative then for the mathematical consistency of equation (4.4), c_i must necessarily be zero, which implies the stability of the system. Hence the theorem.

THEOREM 4.2 : *The perturbations which travel only in the basic flow direction are stable provided that the conditions*

$$DS_0 > 0 \text{ and } J_s \leqslant 1/4$$

hold everywhere in the flow domain.

Proof: Let DS_0 be everywhere positive and the perturbations travel in the basic flow direction so that $\beta=0$. Equation (4.4) then reduces to

$$c_i \left\{ \int_{-1}^{1} \frac{|\phi'|^2 + K^2 |\phi|^2}{Q} dz + \int_{-1}^{1} \frac{U'^2}{Q|U-c|^2} (1/4 - J_s) |\phi|^2 dz \right\} = 0 \qquad \dots (4.5)$$

where

$$J_s = \frac{F_1}{U'^2} = \frac{gd\, DS_0}{c_v\, U^{*2}\, U'^2}\,.$$

It follows that if $J_s \leqslant 1/4$ everywhere in the flow domain, the quantity within the brackets of equation (4.5) is positive definite and for the consistency of this equation, we must necessarily have $c_i=0$, showing thereby, the stability of the system. This proves the assertion.

Cor : For a disturbance (α,β) which travels in an arbitrary direction in the $x-y$ plane, a sufficient condition for stability, though not as powerful as obtained in theorem 4.2, can easily be proved and this really provides the stability for the wave numbers satisfying

$$J_s \leqslant 1/4 \operatorname{Cos}^2\theta$$

where $\alpha = k \operatorname{Cos} \theta$.

Remark 1 : Above theorem gives a feeling that the perturbations which are transverse to the basic flow direction may be most sensitive to instability. Such a situation occurs in many physical contexts and needs a separate treatment.

Remark 2 : It is to be remarked that in addition to the semi-circle obtained in Part-A, Howard's semi-circle can be recovered here also by using Howard's methods.

5. **PART-C Supersonic Disturbances :** For supersonic disturbances, already defined in Part-B, equations (2.7) and (2.8) reduce to

$$V_s^2\, DG = -i\alpha\, \mathfrak{Q}\, P, \qquad \dots (5.1)$$

and

$$\alpha\, \mathfrak{Q}\, DP = i[F - \alpha^2 (U-c)^2]\, G. \qquad \dots (5.2)$$

After eliminating P from these equations, we obtain one equation in G alone, namely,

$$-D\left[\frac{V_s^2}{i\alpha\mathfrak{Q}} DG\right] = \frac{i}{\alpha\mathfrak{Q}} [F_1 - \alpha^2(U-c)^2]\, G. \qquad \dots (5.3)$$

This is the final stability governing equation for supersonic disturbances.

THEOREM 5.1 : *If the entropy gradient DS_0 is everywhere positive then the system is stable in the wave number range given by*

$$\alpha^2 < (F_1/U^2)_{min}.$$

Proof: Multiply equation (5.3) by G^*, the complex conjugate of G, and integrate over the range of z. This yields

$$\int(V_s^2/\mathfrak{Q})\,|\,DG\,|^2\,dz+\int(1/\alpha\mathfrak{Q})\,\{F_1-\alpha^2(U-c)^2\}\,|\,G\,|^2\,dz=0. \quad \ldots \quad (5.4)$$

Now multiplying this equation by c^*, the complex conjugate of c and separating the imaginary part of the resulting equation, we obtain

$$c_i\{\int(V_s^2/\mathfrak{Q})\,|\,DG\,|^2 dz+\int(1/\mathfrak{Q}\alpha)\,\{F_1-\alpha^2\,U^2+\alpha^2\,|\,c\,|^2\}\,|\,G\,|^2 dz=0 \quad \ldots \quad (5.5)$$

It follows from here that if DS_0 is everywhere positive and $\alpha^2 < (F_1/U^2)_{min}$ then for the consistency of equation (5.5) we must necessarily have $c_i=0$ which implies the stability of the system for this wave number range.

THEOREM 5.2 : *If DS_0 is everywhere positive, then the modes are oscillatory in the wave number range given by $\alpha^2 < (F_1/U^2)_{min}$.*

Proof: Proof immediately follows from equation (5.4) with the help of theorem 5.1.

Bounds of c_r and c_i for unstable modes : Let the modes be unstable so that $c_i \neq 0$. Then for the consistency of equation (5.5), we must necessarily have

$$|\,c\,|^2 < U^2 - F_1/\alpha^2 \text{ for some } z.$$

Thus given and unstable wave number α_0 (say) its wave velocity c lies within the circle

$$|\,c\,|^2 < \underset{max}{U^2} - (F_1/\alpha_0^2)_{min}\,.$$

Though this result is not much powerful because of the fact that different bounds are provided for different unstable modes, its importance lies in the fact that bounds for c do depend upon various physical quantities such as entropy gradient and gravity etc. upon which the dependence is expected.

6. Part-D Stability of the fluid at rest : The case, when an inviscid and compressible fluid at rest is confined between two horizontal boundaries, is of physical importance. The equation (3.3) now takes the simple form

$$\int\Big[\frac{F_1\,|\,G\,|^2}{Q}+\frac{\alpha\,Q\,|\,P\,|^2}{v_s^2}\Big]\,dz=\int\Big[\frac{Q\,k^2\,|\,P\,|^2}{\alpha\,|\,c\,|^4}+\frac{\alpha\,|\,G\,|^2}{Q}\Big]\,c^{*2}\,dz \quad \ldots \quad (6.1)$$

The following results are proved in this section :

THEOREM 6.1. *c^2 is real.*

Proof : The imaginary part of equation (6.1) gives

$$\int_{-1}^{+1} \left[\frac{QK^2 \ |P|^2}{\alpha \ |c|^4}+\frac{\alpha \ |G|^2}{Q}\right] c_r \, c_i \, dz = 0 \qquad \text{... (6.2)}$$

which shows that either $c_r = 0$ or $c_i = 0$. In both these cases, c^2 is real. Hence the theorem.

Note : What we have shown is that both c_r and c_i cannot be simultaneously non-zero. Of course, both may be zero as permitted by equation (6.2). Now if $c^2 > 0$, the system will be stable while $c^2 < 0$ will imply the instability of the system.

THEOREM 6.2. *If the entropy gradient is everywhere positive, the system is stable.*

Proof : Taking the real part of equation (6.1), we get

$$\int_{-1}^{1} \left[\frac{F_1 \ |G|^2}{\alpha \ Q}+\frac{\alpha \ Q \ |P|^2}{\nu_s^2}\right] dz$$

$$=\int_{-1}^{1} \left[\frac{k^2 \ |P|^2}{\alpha \ |c|^4}+\frac{\alpha \ |G|^2}{Q}\right] (c_r^2-c_i^2) \, dz \qquad \text{... (6.3)}$$

Let the entropy gradient be everywhere positive. Then equation (6.3) is inconsistent if $c_r = 0$. Therefore c_r cannot be zero in this case. But then according to theorem (6.1), c_i should be zero. Hence this proves the stability of the system.

THEOREM 6.3. *If the entropy gradient DS_0 is everywhere negative the system will be unstable.*

Proof : Multiply equation (3.1) through out by G^*, the complex conjugate of G and integrate within the limits of z. On putting $U=0$ in the resulting equation, we must necessarily have

$$|G|^2 \alpha^3 c^4-(F_1 \ |G|^2+k^2 \ \nu_s^2 \ |G|^2+|DG|^2)\alpha c^2 \ \nu_s^2+ \frac{F_1 \ k^2 \ \nu_s^2}{a} \ |G|^2 = 0$$

at some point z_0.

Let the roots of the above equation be c_1^2, c_2^2.

then $c_1^2\, c_2^2 = \dfrac{F_1\, k^2\, v_s^2}{\alpha^4} < 0,$

as DS_0 and hence F_1 is negative everywhere.

Therefore, one of c_1^2 and c_2^2 should be positive which ensures the instability of the system.

Conclusion : The paper investigates the effects of entropy variations on the stability of an inviscid, compressible and thermally non-conducting fluid of variable density. The problem is first investigated in general and then some particular cases of physical significance are discussed. The conditions are obtained under which the modes cannot be non-oscillatory and further if the non-oscillatory modes are to exist, it is shown that these modes are stable. A circle in $c_r - c_i$ plane with centre at origin and radius U_{max} is obtained which arrests the complex wave velocity of an arbitrary unstable mode. The discussion of subsonic disturbances has revealed that if the entropy gradient is everywhere negative, the system is stable and further the stability of such disturbances is also guaranteed for positive entropy gradients if $J_s < 1/4$ everywhere when the perturbations travel only in the flow direction. The supersonic disturbances appear to be less stable than subsonic disturbances and the stability conditions are obtained. The case, when the fluid is initially at rest, is quite interesting. c^2 is shown to be real and it is shown that the system is stable or unstable according as the entropy gradient is everywhere positive or negative.

References

Blumen, W. (1970) : *J. Fluid Mech.* **40**, 769.

Blumen, W., Drazin P. G. and Billings, D. F. (1975) : *J. Fluid Mech.* **71**, 305.

Case, K. M. (1960) : *Phys. Fluids* **3**, 149.

Chimonas, G. (1970) : *J. Fluid Mech.* **43**, 833.

Drazin P. G. (1958) : *J. Fluid Mech.* **4**, 214.

Drazin, P. G. and Davey, A. (1977) : *J. Fluid Mech.* **82**, 255.

Dyson, F. J. (1960) : *Phys. Fluids.*

Eckart, C. (1963) : *Phys. Fluid.* **6**, 1042.

Howard, L. N. (1961) : *J. Fluid Mech.* **10**, 509.

——————— (1973) : Studies Appl. Math. **52**, 39.

Kelvin, Lord (1910) : Hydrodynamics and general dynamics in mathematical and physical papers (Cambridge, England).

Lalas, D. P. (1975) : *J. Fiuid Mech.* **69**, 65,

Miles, J. W. (1957) : *J. Fluid Mech.* **3**, 185.
Miles, J. W. (1959) : *J. Fluid Mech.*, 583.
Miles, J. W. (1961) : *J. Fluid Mech.* **10**, 496.
Rayleigh, Lord (1900) : Scientific papers (Cambridge, England).
Saini, J. (1974) : Ph.D. Thesis, Meerut University, Meerut.
Warren, F. W. G. (1968) : *J. Mech. Appl. Math.* Vol XXI, 433.

DEPARTMENT OF MATHEMATICS,
INSTITUTE OF ADVANCED STUDIES,
MEERUT UNIVERSITY, MEERUT, INDIA.

Bull. Cal. Math. Soc., 75, 129-135 (1983)

OSCILLATION OF A BODY OF REVOLUTION IN A VISCOUS FLUID

ANADI KUMAR CHATTERJEE and SUKOMAL CHANDRA ROY

(*Received 13 December 1978 ; Revision Received 9 June 1979*)

Abstract : The unsteady boundary layer flow generated in an incompressible homogeneous viscous fluid due to the non-torsional oscillation of a body of revolution is studied. The analysis is carried out for the high frequency oscillation case to investigate various qualitative and quantitative information about the steady and unsteady velocity distributions. Using the method of separation of the steady and unsteady parts of the variables an exact solution is obtained. The concept of the generation of the second boundary layer due to steady streaming is used to determine the structure of the flow field and the associated double boundary layers.

Introduction. Several authors including Stuart (1966), Riley (1965, 1967, 1971), Davidson & Riley (1972), Chatterjee (1971), Chatterjee & Roy (1973, 1978) and others have studied the various aspects of the hydrodynamic unsteady boundary layer flows generated in an imcompressible, homogeneous and viscous fluid due to the non-torsional small amplitude oscillations of cylinders and spheres. Stuart (1966) and Riley (1965) independently made a critical analysis of the possibility of the existence of a second boundary layer of much greater thickness outside the Stokes layer to satisfy all the boundary conditions when the frequency of oscillation is high. Davidson and Riley (1962) experimentally verified the existence of the second boundary layer and the jet like (Beakley-Jet) formation of the steady streaming in the case of a cylinder.

To make an extension of the analysis in the case of a body of revolution, we consider here the high frequency oscillation of a body of revolution with a circular frequency w in an incompressible homogeneous viscous fluid treating the case as an axisymmetric one. The analysis is carried out to determine the steady and unsteady flow-fields and the structures of the associated boundary layers. The method of solution is the use of the method of separation of steady and unsteady parts of each variables, obtaining separate equations for the steady and unsteady velocity components and then using Fetti's method of solution to get an exact solution of the problem.

Mathematical Formulations : The following assumptions are made—

(1) The fluid is incompressible and homogeneous

(2) The Kinematic viscosity is constant

(3) The velocity of oscillation is $U = U_0 \ Cos \ wt$.

(4) The body as well as the flow generated are axisymmetric

(5) The amplitude of oscillation is small compared with the dimension of the body.

The problem remains the same if we fix our co-ordinate system on the body and regard the fluid as oscillating.

The Navier-Stokes equation, when pressure is eliminated, can be written in terms of vorticity in the form

$$\frac{\partial \zeta'}{\partial t}-[\nabla_{\wedge} q'_{\wedge}\zeta']=-\nu\nabla_{\wedge}\nabla_{\wedge}\zeta' \qquad \ldots \quad (1)$$

and

$$\nabla . q'=0 \qquad \ldots \quad (2)$$

where, q' is the velocity vector and $\zeta'=\nabla_{\wedge} q'$ is the vorticity vector.

Now, separating the steady and unsteady parts of the variable q' and ζ' and denoting them by bars and tilds respectively we get

$$\frac{\partial \tilde{\zeta}'}{\partial t}-\nabla_{\wedge}\tilde{q}'_{\wedge}\bar{\zeta}'-\nabla_{\wedge}\bar{q}'_{\wedge}\tilde{\zeta}'-[\nabla_{\wedge}\tilde{q}'_{\wedge}\tilde{\zeta}']_u=-\nu\nabla_{\wedge}\nabla_{\wedge}\tilde{\zeta}' \qquad \ldots \quad (3)$$

$$-\nabla_{\wedge}\bar{q}'_{\wedge}\bar{\zeta}'-[\nabla_{\wedge}\tilde{q}'_{\wedge}\tilde{\zeta}']_s=-\nu\nabla_{\wedge}\nabla_{\wedge}\bar{\zeta}' \qquad \ldots \quad (4)$$

$$\nabla . \tilde{q}'=0 \qquad \ldots \quad (5)$$

$$\nabla . \bar{q}'=0 \qquad \ldots \quad (6)$$

where $[\quad]_u$ and $[\quad]_s$ denote the unsteady and steady parts of the products respectively.

The boundary conditions are that the velocities approach a prescribed oscillatory flow at infinity and zero on the solid boundary.

To reduce these equations to dimensionless form we normalise the velocity by U_0 time by $1/\omega$ and length by l.

Dropping the primes the equations reduce to

$$\frac{\partial \tilde{\zeta}}{\partial t}-\frac{1}{S}[\nabla_{\wedge}\tilde{q}_{\wedge}\bar{\zeta}+\nabla_{\wedge}\bar{q}_{\wedge}\tilde{\zeta}]-\frac{1}{S}[\nabla_{\wedge}\tilde{q}_{\wedge}\tilde{\zeta}]_u=\frac{1}{RS}\nabla_{\wedge}\nabla_{\wedge}\tilde{\zeta} \qquad \ldots \quad (7)$$

$$[\nabla_{\wedge}\bar{q}_{\wedge}\bar{\zeta}]+[\nabla_{\wedge}\tilde{q}_{\wedge}\tilde{\zeta}]_s=\frac{1}{R}\nabla_{\wedge}\nabla_{\wedge}\bar{\zeta} \qquad \ldots \quad (8)$$

$$\nabla . \tilde{q}=0 \qquad \ldots \quad (9)$$

$$\nabla . \bar{q}=0 \qquad \ldots \quad (10)$$

where $R=\frac{U_0 l}{\nu}$ and $S=\frac{l\omega}{U_0}$

Solution : Considering the flow field to be an axisymmetric one and taking the coordinate system as x being parallel to the meridian from a fixed point and y being normal to the surface of the body the equation of continuity becomes (Ghosh 1960).

$$\frac{\partial}{\partial x}(h_2 h_3 u)+\frac{\partial}{\partial y}(h_1 h_3 v)=0 \qquad \dots (11)$$

where $h_1=\dfrac{\mathscr{R}+y}{\mathscr{R}}$, $h_2=1$ and $h_3=r(x,y)$

also $ds^2=h_1^2dx^2+h_2^2dy^2+h_3^2d\varphi^2$ (φ being the meridian angle).

then eqn. (11) becomes $\dfrac{\partial}{\partial x}(ru)+\dfrac{\partial}{\partial y}\left(r.\dfrac{\mathscr{R}+y}{\mathscr{R}}.v\right)=0$ $\qquad \dots (12)$

Now, introducing the stream function Ψ such that

$$h_3 u=-\frac{1}{h_2}\frac{\partial \Psi}{\partial y} \text{ and } h_3 v=\frac{1}{h_1}\frac{\partial \Psi}{\partial x}$$

or
$$u=\frac{1}{r}\frac{\partial \Psi}{\partial y} \text{ and } v=-\frac{\mathscr{R}}{r(\mathscr{R}+y)}\frac{\partial \Psi}{\partial x}$$

then ζ becomes $\quad \zeta=\dfrac{\mathscr{R}}{\mathscr{R}+y}\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}-\dfrac{u}{\mathscr{R}+y}$

We take $r=\rho(x)+y$ so that $r_x=\rho_x$ and $r_y=1$.

We assume that $S=\dfrac{\omega l}{U_0}=\dfrac{1}{\epsilon^2}$, where ϵ is a small quantity and R is of the order much greater than one.

Since the flow field is generated due to the imposed oscillation we consider eqn. (7) first. The above assumption on R and S enables us to conceive of the Stokes layer in which $\dfrac{\partial \tilde{\zeta}}{\partial t}$ interacts with $\dfrac{1}{RS}\nabla_\wedge\nabla_\wedge\tilde{\zeta}$.

i.e.
$$u\frac{\partial \tilde{\zeta}}{\partial t}=\frac{1}{RS}\nabla^2\tilde{\zeta}=\frac{1}{RS}\left[\frac{\partial^2\tilde{\zeta}}{\partial y^2}+\frac{1}{(\mathscr{R}+y)}\frac{\partial \tilde{\zeta}}{\partial y}+\frac{1}{r}\frac{\partial \tilde{\zeta}}{\partial y}+\frac{R^2}{(\mathscr{R}+y)^2}\frac{\partial^2\tilde{\zeta}}{\partial x^2}\right]$$

Putting $y=k\eta$ we get

$$\frac{\partial \tilde{\zeta}}{\partial t}=\frac{1}{Rs}\left[\frac{1}{k^2}\frac{\partial^2\tilde{\zeta}}{\partial \eta^2}+\frac{1}{k(\mathscr{R}+y)}\frac{\partial \tilde{\zeta}}{\partial \eta}+\frac{1}{rk}\frac{\partial \tilde{\zeta}}{\partial \eta}+\frac{\mathscr{R}^2}{(\mathscr{R}+y)^2}\frac{\partial \tilde{\zeta}}{\partial x^2}\right]$$

i.e.
$$K^2Rs=1 \text{ or } K^2=\frac{1}{Rs} \quad \therefore \quad K=\frac{1}{\sqrt{Rs}}$$

$$\therefore \quad \frac{\partial \tilde{\zeta}}{\partial t} = \frac{\partial^2 \tilde{\zeta}}{\partial \eta^2} \qquad \cdots \quad (13)$$

Expanding $\tilde{\Psi}$ in powers of ϵ in the way

$$\tilde{\Psi} = \tilde{\Psi}_0 + \epsilon \tilde{\Psi}_1 + \epsilon^2 \tilde{\Psi}_2 + \ldots\ldots$$

equation (13) becomes

$$\frac{\partial^3 \tilde{\Psi}_0}{\partial \eta^2 \partial t} = \frac{\partial^4 \tilde{\Psi}_0}{\partial \eta^4} \qquad \cdots \quad (14)$$

whose solution satisfying the boundary conditions

$$\tilde{\Psi}_{0\eta} = \tilde{\Psi}_{0x} = 0 \text{ on } \eta = 0 \text{ and } \tilde{\Psi}_{0\eta} = \rho \sin x \,.\, e^{it},\ \tilde{\Psi}_{0x} = \rho \cos x \,.\, e^{it} \text{ as } \eta \to \infty.$$

$$\text{i. e. } \tilde{\Psi}_0 = \epsilon\, e^{it} \sin x \,.\, \rho \left[-\frac{(1-i)}{\sqrt{2}} + \eta + \frac{(1-i)}{\sqrt{2}}\, e^{\frac{-(1+i)}{\sqrt{2}}\eta} \right] \qquad \cdots \quad (15)$$

This implies that the series of $\tilde{\Psi}$ in powers of ϵ will be

$$\tilde{\Psi} = \epsilon [\tilde{\Psi}_0 + \epsilon\, \tilde{\Psi}_1 + \epsilon^2\, \tilde{\Psi}_2 + \ldots]$$

whence

$$\tilde{\Psi}_0 = e^{it} \sin x.\, \rho \left[\frac{-(1-i)}{\sqrt{2}} + \eta + \frac{(1-i)}{\sqrt{2}}\, e^{-\frac{(1+i)}{\sqrt{2}}\eta} \right] \qquad \cdots \quad (16)$$

Once the unsteady component is obtained inside the boundary layers, the nature of the steady component generated due to the interaction of the unsteady components is examined from equation (8). The term representing the generation of the steady component in the L.H.S. of eqn. (8) is the second term $[\nabla_\wedge \tilde{q}_\wedge \tilde{\zeta}]_S$. In order that this term can interact with the term $\frac{1}{R}\nabla^2 \bar{\zeta}$ one must have

$$\bar{\Psi} = \epsilon^3 [\bar{\Psi}_0 + \epsilon \bar{\Psi}_1 + \epsilon^2 \bar{\Psi}_2 + \ldots\ldots]$$

Then

$$\frac{\partial^4 \bar{\Psi}_0}{\partial \eta^4} = \frac{\sin 2x}{2} \alpha^{1/2} \rho \,.\, (\sqrt{2}\, E - \sqrt{2}\, e^{-\sqrt{2}\eta} + i\eta E)$$

$$- \frac{\sin^2 x}{2\sqrt{2}} \alpha^{1/2} \frac{\partial r}{\partial x} \; [(1+i)\, E - e^{-\sqrt{2}\eta}] \qquad \cdots \quad (17)$$

where $E = e^{\frac{-(1+i)}{\sqrt{2}}\eta}$

Its solution satisfying the b.c. $\bar{\Psi}_{0\eta}=0$, $\bar{\Psi}_{0x}=0$ on $\eta=0$

and $\bar{\Psi}_{0\eta}\to 0$, $\bar{\Psi}_{0x}\to 0$ as $\eta\to\infty$ is

$$\begin{aligned}\bar{\Psi}_0=&\Big[\frac{(13+8i)}{4\sqrt{2}}\ \alpha^{1/2}\,\rho\sin 2x-\frac{(5+4i)}{8\sqrt{2}}\ \alpha^{1/2}\left(\frac{\partial r}{\partial x}\right)_{\eta=0}\sin^2 x\Big]\\&+\Big[-\rho\alpha^{1/2}\,\frac{(3+8i)}{4}\sin 2x+\alpha^{1/2}\,\frac{(1+4i)}{8}\left(\frac{\partial r}{\partial x}\right)_{\eta=0}\sin^2 x\Big]\eta\\&+\frac{\sin 2x}{2}\,\alpha^{1/2}\,\rho\Big[-\sqrt{2}E-\frac{e^{-\sqrt{2}\eta}}{2\sqrt{2}}-iE\Big(\eta+\frac{4\sqrt{2}}{(1+i)}\Big)\Big]\\&+\frac{\sin^2 x}{2\sqrt{2}}\,\alpha^{1/2}\left(\frac{\partial r}{\partial x}\right)\Big[(1+i)\,E+\frac{e^{-\sqrt{2}\eta}}{4}\Big]\end{aligned}\qquad\ldots\ (18)$$

On examination of equations (15) and (18) it is seen that $\tilde{\Psi}_0$ in (15) satisfies the boundary conditions at $\eta=0$ and $\eta\to\infty$ whereas $\bar{\Psi}_0$ in (18) satisfies the b.c. at $\eta=0$ but not at $\eta\to\infty$ i.e. $\bar{\Psi}_{0\eta}$ does not tend to zero as $\eta\to\infty$. Rather it becomes

$$\bar{\Psi}_{0\eta}=\Big[-\rho\alpha^{1/2}\,\frac{(3+8i)}{4}\sin 2x+\alpha^{1/2}\,\frac{(1+4i)}{8}\left(\frac{\partial r}{\partial x}\right)_{\eta=0}\sin^2 x\Big]\qquad\ldots\ (19)$$

If this steady velocity distribution at the outer edge of the boundary layer were absent, the flow outside the boundary layer becomes Stokes flow.

Since the investigation of the steady velocity was started from eqn. (8), let us re-examine eqn. (8). It is seen that the second term on L.H.S. $[\nabla_\wedge\tilde{q}_\wedge\tilde{\zeta}]_s$ tends to zero at the outer edge of the boundary layer but as the steady velocity is of $0(\epsilon^2)$ the first term on L.H.S. $[\nabla_\wedge\bar{q}_\wedge\bar{\zeta}]$ representing the effect of the non-linear Reynolds Stresses, interacts with the viscous term $\frac{1}{R}\ \nabla^2\bar{\zeta}$.

For their interaction to occur Stuart has shown that $R_s=\frac{R}{\delta}$ must be much greater than one i.e. $R_s >> 1$ for the flow to be again of the boundary layer type.

The problem now turns out to be one that we are to determine the nature of the steady velocity distribution in the outer layer due to a steady velocity distribution of the order of $0(\epsilon^2)$ at the edge of the inner layer. That means the boundary conditions of the outer layer problem are that there is a steady velocity distribution of $0(\epsilon^2)$ at $\bar{\eta}=0$ and steady velocity tends to zero as $\bar{\eta}\to\infty$.

Outer Layer Solution : It is assumed that $y=k_1\bar{\eta}$ and $\bar{\Psi}=k_2(\bar{\Psi}_0+\epsilon\bar{\Psi}_1+\epsilon^2\bar{\Psi}_2+\ldots)$ is the steady stream function in the outer layer. The governing equation is

$$\nabla_\wedge\bar{q}_\wedge\bar{\zeta}=\frac{1}{R}\ \nabla^2\bar{\zeta}.\qquad\ldots\ (20)$$

which reduces to (considering only the $\bar{\Psi}_0$ term)

$$\frac{\partial(\bar{\Psi}_0, \bar{\Psi}^{A}_{0\eta\eta})}{\partial(x,\bar{\eta})} + \frac{1}{\rho} \frac{\partial r}{\partial x} \frac{\partial \bar{\Psi}_0}{\partial \bar{\eta}} \frac{\partial^2 \bar{\Psi}}{\partial \bar{\eta}^2} = -\rho \frac{\partial^4 \bar{\Psi}_0}{\partial \bar{\eta}^4} \qquad \ldots \quad (21)$$

where $RK_1K_2=1$.

Now Fetti's method of solution is used. (The method may be regarded either as a formal expansion scheme in which a small parameter δ is first introduced and then set equal to unity or as an iterative scheme. It is also worthy of mention that Fetti's method is remarkably easy to apply when the boundary layer accurately has a behaviour like e^{-Kz} where K is a constant or a function of x, as $z \to \infty$, for then the analysis involves only elementary operations on exponential functions of this kind). Then

$$\bar{\Psi}_0 = \xi(x) + \delta \bar{\Psi}_{00} + \delta^2 \bar{\Psi}_{01} + \cdots\cdots$$

where δ is a parameter, which later we shall set equal to unity. $\bar{\Psi}_{00}$, $\bar{\Psi}_{01}$, $\bar{\Psi}_{02}$ etc. are functions of both $\bar{\eta}$ and x while $\xi(x)$ is a function of x only and is independent of $\bar{\eta}$.

Then from eqn. (22) we get

$$\frac{\partial^4 \bar{\Psi}_{00}}{\partial \bar{\eta}^4} + \frac{\xi'(x)}{\rho} \frac{\partial^3 \bar{\Psi}_{00}}{\partial \bar{\eta}^3} = 0 \qquad \ldots \quad (22)$$

Its solution $\bar{\Psi}_{00}$ satisfying the boundary condition is

$$\bar{\Psi}_{00} = \frac{\rho}{[\xi'(x)]} \left[\rho \alpha^{1/2} \frac{(3+8i)}{4} \sin 2x - \left(\frac{\partial r}{\partial x} \right)_{\bar{\eta}=0} \alpha^{1/2} \sin^2 x \frac{(1+4i)}{8} \right] e^{-\frac{\xi'(x)\bar{\eta}}{\rho}} \qquad \ldots \quad (23)$$

But, since $[\bar{\Psi}_{00}]_{\bar{\eta}=0} = -\xi(x)$

whence

$$\xi(x) = \left[\frac{1}{4} \rho \alpha^{1/2} (3+8i) \cos 2x + \frac{(1+4i)}{4} \alpha^{1/2} \left(\frac{\partial r}{\partial x} \right)_{\bar{\eta}=0} \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) \right]^{1/2} \rho^{1/2} \qquad \ldots \quad (24)$$

Thus eqn. (23) with (24) showing the value of $\xi(x)$ is the solution for $\bar{\Psi}_{00}$ in the outer layer. Thus the steady velocity generated at the outer edge of the inner boundary layer decays to zero in the outer layer, of thickness$=1/\epsilon$ times the inner layer thickness. Once the $\tilde{\Psi}_0$, $\bar{\Psi}_0$ are completely determined in the inner layer and $\bar{\Psi}_0$ in the outer layer, the other terms $\tilde{\Psi}_1$, $\bar{\Psi}_1$, $\tilde{\Psi}_2$, $\bar{\Psi}_2$ etc. can easily be calculated.

Conclusion : 1. An exact solution of the problem is obtained.

2. The method of separation of steady and unsteady parts shows when the interaction begins to generate steady flow from purely unsteady components and also gives an equation for the steady flow.

Acknowledgement : The work was carried out under the guidance of Prof. Ambarish Ghosh. The authors are thankful to Prof. N. Riley of the University of East Anglia, Norwich, Great Britain, for his helps.

References

Chatterjee, A. K. (1971) : *Ind. J. Phys*. **45**, 553.

Chatterjee, A K. & Roy, S. (1973) : 39th Annual. Conf. *I.M.S.* Jadavpur.

Chatterjee, A. K. & Roy, S. (1978) : To appear in *Bull. Cal. Math. Soc.*

Davidson, B. J. & Riley, N. (1972) : *J.F.M.* **53**, 287.

Ghosh, A. (1960) : Thesis-Univ. D'AIX-MARSEILLE.

Riley, N. (1965) : *Mathematica* **12**, 161.

Riley, N. (1967) : *J. Inst. Maths. Applies* **3**, 419.

Riley, N. (1971) : *ZAMP* **22**, 645.

Stuart, J. T. (1966) : *J.F.M.* **24**, 673.

HYDRAULIC STUDY DEPARTMENT
CALCUTTA PORT TRUST, CALCUTTA.

RIVER RESEARCH INSTITUTE,
CALCUTTA.

Bull. Cal. Math. Soc. 75, 137-143 (1983)

UNSTEADY CHANNEL FLOW OF A CONDUCTING FLUID WITH SUSPENDED PARTICLES

S. C. GUPTA

(*Received 13 September 1978*)

Abstract : In this paper, the problem of the flow of a viscous incompressible fluid embedded with small spherical particles in the presence of a transverse magnetic field in a channel has been discussed. The cross section of the channel is equilateral trangle and the walls are non-conducting. The analysis applies to flows with pressure gradients which are arbitrary functions of time. Explicit expressions for the exact velocities of the fluid and particles are obtained by using Laplace and equitriangular transforms. It has been found that the velocity of fluid is greater than that of particles when the flow takes place under the influence of a constant pressure gradient. The velocity of fluid and particle decrease with increase in the intensity of the magnetic field.

Nomenclature :

H_0	Strength of the imposed magnetic field
k	$N_0 m/\rho$ density ratio of particles to fluid (per unit volume of flow field)
τ	K/ν particle relaxation time
N_0	the number density of particles
m	the mass of a particle
K	the Stokes resistance coefficient
M	$\mu_e^2 H_0^2/\rho$
t	time
u	gas velocity
v	particle velocity
μ_e	magnetic permeability of the gas
ν	kinematic viscosity of the gas
σ	electric conductivity of the gas
ρ	density of the gas
p	pressure

Introduction : The unsteady flow of a viscous incompressible fluid in a duct of equitriangular cross-section under the influence of an exponential pressure gradient has been discussed by Sen (1964). In recent years a number of studies of fluid embedded with particles [References] have appeared in literature. The study of fluid or gas having uniform distribution of solid particles is of interest in a wide range of technical problems. These include fluidization, environmental pollution, combustion and more recently blood flow.

Rao (1969) has discussed the flow of a dusty gas in a circular cylinder under the influence of a pressure gradient which varies exponentially with time. Lu and Miller (1967) have discussed the flow of the dusty gas under the influence of a constant pressure gradient in a channel with equitriangular cross-section. Singh and Pathak (1977) have considered the flow of the dusty viscous fluid in a tube with sector of a circle as cross-section under the influence of exponential pressure gradient. Recently, Gupta and Gupta (1976) have studied the flow of the dusty gas in a rectangular channel with arbitrary time varying pressure gradient. The flow induced in an incompressible dusty gas by an infinite flat plate set into motion in its place by oscillation and impulse, in the presence of a transverse magnetic field, has been studied by Healy and Yang (1973).

The present communication reports the flow of an incompressible viscous electrically conducting fluid, embedded with small inert particles in the presence of a transverse magnetic field of uniform intensity in a channel whose cross-section is an equilateral triangle and whose walls are non-conducting. It is proposed to develop the general time dependent flow model and to obtain exact explicit expressions for both the gas and particle velocities when the pressure gradient is an arbitrary function of time. The particular case when the pressure gradient is an absolute constant is discussed in detail.

Non-circular ducts are frequently used in automobile radiators, nuclear power plants, aerospace vehicles etc. as they can be fitted within the available space between compactly placed components. The flow of fluid embedded with particles in such ducts is of great importance. One finds that the behaviour of oil or fuel in ducts due to presence of particles is considerably changed and many new phenomena can be observed.

Formulation of the problem : The appropriate equations of motion for the problem (fig. 1) taken from Saffman (1962) and Pai (1962) are

$$\frac{\partial u}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 u+\frac{k}{\tau}(v-u)-Mu, \quad \ldots (1)$$

$$\frac{\partial v}{\partial t}=\frac{1}{\tau}(u-v), \quad \ldots (2)$$

where

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

$$P_1 = \frac{a}{\sqrt{3}} - \frac{\sqrt{3}y}{2} + \frac{x}{2}$$

$$P_2 = \frac{a}{\sqrt{3}} + \frac{3y}{2} + \frac{x}{2}$$

$$P_3 = \frac{a}{\sqrt{3}} - x\ ,$$

$$= \sqrt{3}\, a - (P_1 + P_2)$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$B : P_1 = 0,\ P_2 = 0,\ P_3 = 0$$

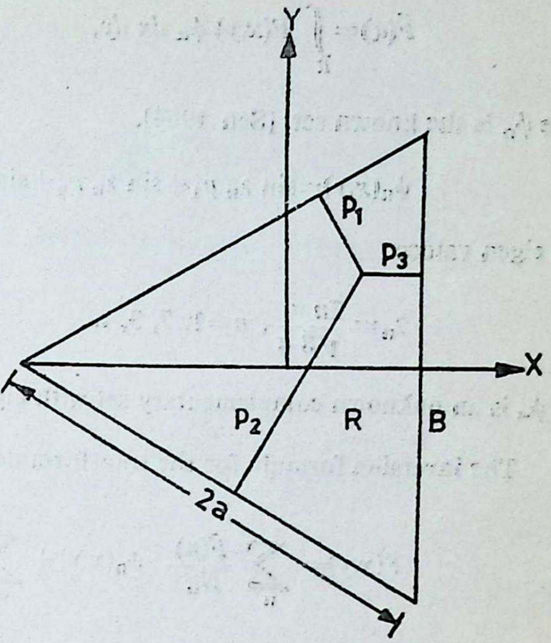


Fig. 1 Equitriangular region

The suspended particles are very small and the Reynolds number of the relative motion of particle and gas is smaller than unity (Saffman 1962). The electromagnetic effect entering the problem in the last term of (1) is an approximation under the suspension of small electric conductivity.

The initial and boundary conditions for the problem are

$$\left.\begin{aligned} u(x,y,0)&=0 \\ v(x,y,0)&=0 \end{aligned}\right\} \qquad \ldots \quad (3)$$

and

$$\left.\begin{aligned} u(x,y,t)&=0 \\ v(x,y,t)&=0 \end{aligned}\right\} \text{ on } B \qquad \ldots \quad (4)$$

For the present geometry it is convenient to solve the problem with the help of equitriangular transform. The equitriangular transform has, recently, been investigated by Lu and Miller [1967] in terms of known eigen functions—ψ_n, discovered by Sen [1964] and unknown complementary set ϕ_v. It is defined as

$$\tilde{F}(n) = \int_R F(x,y)\, \psi_n\, dx\, dy,$$

$$\bar{F}(\alpha)=\int_R F(x,y)\,\phi_\alpha\,dx\,dy, \qquad \ldots (5)$$

where ψ_n is the known set, [Sen 1964],

$$\psi_n(x,y)=\sin\lambda_n p_1+\sin\lambda_n p_2+\sin\lambda_n p_3 \qquad \ldots (6)$$

with eigen values

$$\lambda_n=\frac{2n\pi}{\sqrt{3}\,a},\ n=1,2,3,\ldots \qquad \ldots (7)$$

and ϕ_α is an unknown complementary set with eigen values ξ_α.

The inversion formula for the transform defined by equation (5) is

$$F(x,y)=\sum_n\frac{\tilde{F}(n)}{N_n}\ \psi_n(x,y)+\sum_\alpha\frac{\bar{F}(\alpha)}{M_\alpha}\ \phi_\alpha(x,y), \qquad \ldots (8)$$

where N_n and M_α are the norms of ψ_n and ϕ_α.

In the present investigation the following properties of the equitriangular transform will be used. If $F(x,y)$ vanishes on the boundary B then, Lu & Miller [1967],

$$\nabla^2\tilde{F}=-\lambda_n^2\tilde{F}, \qquad \ldots (9)$$

$$\nabla^2\bar{F}=-\xi_\alpha^2\bar{F}, \qquad \ldots (10)$$

and

$$\tilde{C}=\frac{3\sqrt{3}\,a^2}{n\pi}\,C, \qquad \ldots (11)$$

$$\bar{C}=0,$$

where C is a constant.

Solution of the problem : In order to fully specify the flow field, eqs. (1) and (2) will be solved subject to conditions prescribed by eqs. (3) and (4). To eqs. (1) & (2) apply the first of the transform defined by eq. (5) and use eq. (9) under the condition (4) to get

$$\frac{\partial\tilde{u}}{\partial t}=\frac{3\sqrt{3}\,a^2}{n\pi}f(t)-\nu\lambda_n^2\tilde{u}+\frac{k}{\tau}(\tilde{v}-\tilde{u})-M\tilde{u}, \qquad \ldots (13)$$

$$\frac{\partial\tilde{v}}{\partial t}=\frac{1}{\tau}(\tilde{u}-\tilde{v}), \qquad \ldots (14)$$

where

$$\tilde{u}\,(n,\,t)=\int_R u\,(x,\,y,\,t)\quad \psi_n\,dx\,dy$$

$$\tilde{v}\,(n,\,t)=\int_R v\,(x,\,y,\,t)\quad \psi_n\,dx\,dy,\ \frac{1}{\rho}\,\frac{\partial p}{\partial z}=f\,(t) \qquad \ldots\ (15)$$

$f(t)$ being an arbitrary function of time t. The initial condition (3) transform to

$$\left.\begin{aligned}\tilde{u}\,(n,\,0)&=0\\ \tilde{v}\,(n,\,0)&=0\end{aligned}\right\} \qquad \ldots\ (16)$$

Now apply Laplace transform defined by

$$\tilde{u}_H\,(n,\,s)=\int_0^\infty \tilde{u}\,(n,\,t)\,e^{-st}\,dt$$

$$\tilde{v}_H\,(n,\,s)=\int_0^\infty \tilde{v}\,(n,\,t)\,e^{-st}\,dt \qquad \ldots\ (17)$$

to eqs. (13)–(14) and use the transformed initial condition (16) to get

$$s\,\tilde{u}_H=\frac{3\sqrt{3}\,a^2}{n\,\pi}\,f_H\,(s)-\nu\,\lambda_n^2\,\tilde{u}_H+\frac{k}{\tau}\,(\tilde{v}_H-\tilde{u}_H)-M\tilde{u}_H, \qquad \ldots\ (18)$$

$$s\,\tilde{v}_H=(\tilde{u}_H-\tilde{v}_H), \qquad \ldots\ (19)$$

where $f_H\,(s)$ is the Laplace transform of $f\,(t)$.

Solving eqs. (18) and (19) for u_H and v_H and then inverting the Laplace transform by convolution theorem we get

$$\tilde{u}\,(n,\,t)=\frac{3\sqrt{3}\,a^2}{(D_1-D_2)\,n\,\pi}\left[\int_0^t f\,(t-\eta)\,\left\{(D_1+\tau')\,e^{D_1\eta}-(D_2+\tau')\,e^{D_2\eta}\right\}d\,\eta\right] \qquad \ldots\ (20)$$

$$\tilde{v}\,(n,\,t)=\frac{3\sqrt{3}\,a^2\,\tau'}{n\,(D_1-D_2)\,\pi}\left[\int_0^t f\,(t-\eta)\,\left\{e^{D_1\eta}-e^{D_2\eta}\right\}d\,\eta\right], \qquad \ldots\ (21)$$

where

$$\begin{matrix}D_1\\ D_2\end{matrix}=-\frac{1}{2}\left[(1+k)\,\tau'+(\nu\,\lambda_n^2+M)\mp[\{(1+k)\,\tau'+\nu\,\lambda_n^2+M\}^2-4\,(\nu\,\lambda_n^2+M)]^{\frac{1}{2}}\right] \qquad \ldots\ (22)$$

and $\tau'=1/\tau$.

Proceeding exactly on similar lines it can readily be shown that the second of the transforms, as defined by eq. (5), $\bar{u}$ and $\bar{v}$ of u and v, are zero i.e.

$$\left.\begin{aligned} \bar{u}(\alpha, t)&=0, \quad \cdots (23)\\ \bar{v}(\alpha, t)&=0, \quad \cdots (24)\end{aligned}\right\}$$

In view of inversion formula (8) and eqs. (20)—(23) and (21)—(24) the velocities of the gas and particles are given by

$$u(x, y, t)=\frac{2}{\pi}\sum_{n}\left[\int_{0}^{t} f(t-\eta)\left\{(D_1+\tau')e^{D_1\eta}-(D_2+\tau')e^{D_2\eta}\right\}d\eta\right]\frac{\psi_n}{(D_1-D_2)\,n} \quad \cdots (25)$$

$$v(x, y, t)=\frac{2}{\pi}\sum_{n}\left[\int_{0}^{t} f(t-\eta)\left\{e^{D_1\eta}-e^{D_2\eta}\right\}d\eta\right]\frac{\psi_n}{(D_1-D_2)\,n} \quad \cdots (26)$$

Equations (25) and (26) represent the most general solution in convolution integral form of the velocities of the gas and particles respectively, when the flow takes place from rest under the influence of an arbitrary time varying function sufficiently well behaved for its Laplace transform to exist.

If the masses of particles are small their influence on the fluid is reduced. In the limit as $m\to 0$, the fluid become an ordinary viscous. Thus if we let $m\to 0$ in equation (25) the resulting expression represents the magnetohydrodynamic flow of the viscous incompressible fluid through the channel. The corresponding expression form is given by

$$u(x, y, t)=\frac{2}{\pi}\sum_{n}\left[\int_{0}^{t} f(t-\eta)\,e^{-(\nu\lambda_n^2+M)\eta}\,d\eta\right]\frac{\psi_n}{(\nu\lambda_n^2+M)\,n}\,. \quad \cdots (27)$$

Particular Case : A number of particular cases when the flow takes place due to constant, periodic, exponentially time varying pressure gradient etc. can readily be obtained from eqs. (25)—(26). As an example of the main results, we shall discuss the flow due to a constant pressure gradient C. Substituting $f(t)=C$ in equations (25) and (26), and simplifying, we get

$$u(x, y, t)=\frac{2}{\pi}\sum_{n}\left[1-\frac{(D_1+\nu\lambda_n^2+M)\,e^{D_2 t}-(D_2+\nu\lambda_n^2+M)\,e^{D_1 t}}{D_1-D_2}\right]\times\frac{\psi_n}{(\nu\lambda_n^2+m)\,n} \quad \cdots (28)$$

and

$$v(x, y, t) = \frac{2}{\pi} \sum_{n} \left[1 - \frac{D_1 e^{D_2 t} - D_2 e^{D_1 t}}{D_1 - D_2} \right] \frac{\psi n}{(v \lambda_n^2 + M) n} . \qquad \dots (29)$$

Eqs. (28) and (29) represent the velocities of the fluid and particles under the influence of a constant pressure gradient C through the channel. From these equations it is evident that the fluid moves faster than the particles and with increase in 'M' the velocities of the fluid and particles decrease. In the absence of magnetic fluid i.e. $M=0$, equation (28) and (29) give hydrodynamically unsteady velocities of the fluid and particles under the influence of a constant pressure gradient C, which correspond to the results obtained by Lu and Miller (1967).

References

Dube, S. N. and Srivastava L. P. (1972) : *Def. Sci. J.*, 22, 195.

Gupta, R. K. and Gupta, S. C. (1976) : *ZAMP*, 27, 119.

Healy, J. V. and Yang, H. T. (1972) : *Astronautica Acta*, **17**, 851.

Healy, J. V. and Yang, H. T. (1973) : *Appl. Scientific Res.*, 387.

Liu, J. T. C. (1966) : *Phys. Fluid* **9**, 1716.

Liu, J. T. C. (1972) : *Astronautica Acta* **17**, 851.

Lu, Phu-Chang and Miller, R. W. 1967) : *J. Appl. Mech. Trans-ASME, Ser, E,* **34**, 515.

Pai, S. I. (1962) : Magneto-gas-dynamics and Plasma Dynamics, Prentice Hall, Inc., Englewood cliffs, New Jersey.

Reddy, Y. B. (1972) : *Def. Sci. J.* 22, 149.

Rao, P. S. S. (1969) : *Def. Sci. J.* **19**, 135

Sen, B. (1964) : Revue Roumaine Des' *Sci. Tech. Ser de Mech. Appliques* **9**, 307.

Singh, J. P. and Pathak, R. S. (1977) : *Ind. J. Pure and Appl. Math.* **8**, 314.

Saffman, P. G. (1962) : *J. Fluid Mech.* **13**, 120.

Vimla, C. S. (1972) : *Def. Sci. J.* 22, 231.

DEPARTMENT OF MATHEMATICS,
VARDHAMAN COLLEGE,
BIJNOR-246 701.

Bull. Cal. Math. Soc., **75**, 145-152 (1983)

SCATTERING OF MUONS BY A SCREENED COULOMB POTENTIAL

L. Chatterjee and T. Roy

(*Received 22 September 1978*)

Abstract: Scattering of μ^--mesons has been studied, taking into account the screening by atomic electrons. The Dirac Equation has been solved for a screened Coulomb potential with a cut-off parameter 'a_1', and an expression for the scattering phase shift has been derived. This is shown to go to zero as the potential goes to zero. The parameter 'a_1' may be varied to fit experimental data.

Introduction. Muon scattering experiments require much less radiative corrections than the corresponding electron experiments due to the larger muonic rest mass. Thus high energy muon beams available at the accelerator centres are being increasingly used to study the elastic and inelastic structure of target hadrons and the scaling behaviour of lepton-proton Scattering and also to obtain sensitive tests of Quantum Electrodynamics [Hughes and Kinsohita].

To date no significant differences have been observed in the scattering or general electrodynamic interactions of electrons and muons so that Quantum electrodynamics is considered to be adequate in describing muon dynamics even at high energies and for large momentum transfers by the muon. Thus any theory developed for electrons is expected to hold for the case of muon-scattering and vice versa.

The anomalies associated with electron scattering experiments were successfully resolved by Allis and Morse (1931) by using a simplified atomic model taking screening of the nucleus by atomic electrons into account. They used a screened Coulomb potential with a cut-off parameter 'a_1' of type

$$V(r) = -Ze^2 \left(\frac{1}{r} - \frac{1}{a_1}\right) \qquad \cdots \quad (1)$$

They fitted the experimental data for different values of a_1 to explain the anomalies and tabulated value of a_1 for different elements.

In this communication we have solved the Dirac Equation for a muon acted on by such an Allis-Morse potential $V(r)$ for the two regions $r \leqslant a_1$ and $r \geqslant a_1$ and determined the phase shift from the boundary conditions at $r = a_1$. The phase shift is shown to go to zero as the potential goes to zero and to converge for increasing j values.

We note however that the values of a_1 tabulated by Allis and Morse may not be suitable for the case of muon scattering, so that a_1 should be independently determined from muon scattering data. In fact a_1 can be conveniently used as a parameter to fit the experimental data.

Solutions for $r \leqslant a_1$ and $r \geqslant a_1$.

We write the general solution of the Dirac Equation in the absence of an electromagnetic field

$$[p_0 + \alpha \, . \, p + \beta] \, \Psi_\lambda = 0 \qquad \ldots \quad (2)$$

in the form [Mott and Sneddon 1948]

$$\Psi = \sum_j A_j \begin{pmatrix} f(r) \, T_j^1 \, y_{j+\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ -f(r) \, T_j^2 \, y_{j+\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \\ G(r) \, T_j^3 \, y_{j-\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ G(r) \, T_j^4 \, y_{j-\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} + \sum_j B_j \begin{pmatrix} g(r) \, T_j^3 \, y_{j-\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ g(r) \, T_j^4 \, y_{j-\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \\ F(r) \, T_j^1 \, y_{j+\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ -F(r) \, T_j^2 \, y_{j+\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \qquad \ldots \quad (3)$$

where

$$\left. \begin{aligned} T_j^1 &= \left(\frac{j+m}{2j} \right)^{\frac{1}{2}}, \quad T_j^2 = \left(\frac{i-m}{2j} \right)^{\frac{1}{2}}, \\ T_j^3 &= \left(\frac{j+1-m}{2j+2} \right)^{\frac{1}{2}}, \quad T_j^4 = \left(\frac{j+1+m}{2j+2} \right)^{\frac{1}{2}} \end{aligned} \right\} \qquad \ldots \quad (4)$$

and

$$y_{l\,.\,m}(\theta,\varphi) = (-1)^m \left\{ \frac{(2l+1)!\,(l-m)!}{4\pi(l+m)!} \right\}^{\frac{1}{2}} \, p_l^m \, (\cos\theta) \, e^{im\varphi} \qquad \ldots \quad (5)$$

$$(-l \leqslant m \leqslant l).$$

The functions f, G, F, g satisfy the radial Dirac Equations in two sets.

$$f' + \left(j + \frac{3}{2} \right) \frac{f}{r} + \frac{i}{\hbar c} \, W'_1 G = 0 \qquad \ldots \quad (6)$$

$$G' - \left(j - \frac{1}{2} \right) \frac{G}{r} + \frac{i}{\hbar c} \, W'_2 \, f = 0 \qquad \ldots \quad (7)$$

$$F' + \left(j + \frac{3}{2} \right) \frac{F}{r} + \frac{i}{\hbar c} \, W'_2 \, g = 0 \qquad \ldots \quad (8)$$

$$g'-\left(j-\frac{1}{2}\right)\frac{g}{r}+\frac{i}{\hbar c}W'_1F=0 \qquad \ldots (9)$$

where

$$\left.\begin{aligned} W'_1&=W'+\frac{Ze^2}{r}-mc^2,\ W'_2=W'+\frac{Ze^2}{r}+mc^2 \\ W'&=W-\frac{Ze^2}{a_1} \qquad \text{for } r\leqslant a_1 \end{aligned}\right\} \qquad \ldots (10)$$

and

and

$$W'_1=W-mc^2=W_1 \text{ and } W'_2=W+mc^2=W_2 \text{ for } r\geqslant a_1 \qquad \ldots (11)$$

(since W' is then equal to W).

For free particles

$$W^2>m^2c^4.$$

It is seen that the set (6,7) transform into (8,9) when j is replaced by $-j-1$, and f by g & F by G.

We consider first the region $r\geqslant a_1$ (6) & (7) then become

$$\frac{df}{dr}+\left(j+\frac{3}{2}\right)\frac{f}{r}+\frac{i}{\hbar c}W_1G=0 \qquad \ldots (12)$$

$$\frac{dG}{dr}-\left(j-\frac{1}{2}\right)\frac{G}{r}+\frac{i}{\hbar c}W_2f=0 \qquad \ldots (13)$$

Differentiating (12) with respect to r and simplifying we have finally

$$\frac{d^2f}{d\xi^2}+\frac{2}{\xi}\frac{df}{d\xi}+\left[1-\frac{\left(j+\frac{1}{2}\right)\left(j+\frac{1}{2}+1\right)}{\xi^2}\right]f=0 \qquad \ldots (14)$$

where

$$\xi=p_0r \text{ with } p_0^2=W_1W_2/\hbar^2c^2.$$

If we change the dependent variable to $y=f\sqrt{\xi}$, the solution of (14) can be written as

$$f(\xi)=\frac{C_1}{\sqrt{\xi}}J_n(\xi)+\frac{C_2}{\sqrt{\xi}}y_n(\xi) \qquad \ldots (15)$$

$$G(\xi)=\frac{i\hbar cp}{\sqrt{\xi}(W-mc^2)}\left[C_1J_{n-1}(\xi)+C_2y_{n-1}(\xi)\right] \text{(using (12)).} \qquad \ldots (16)$$

we now find the radial functions for $r\leqslant a_1$.

Referring to (6) and (7),

We make the following substitutions $\zeta=ze^2/\hbar c$,

$$ip=\frac{mc}{\hbar}\sqrt{1-\epsilon^2} \text{ or } p=\frac{mc}{\hbar}\sqrt{\epsilon^2-1}\ ,\quad \rho=2ipr,$$

$$f(r)=\sqrt{\epsilon-1}\,e^{-\rho/2}\rho^{\gamma-1}(f_1-f_2) \qquad \ldots (17)$$

$$G(r)=\sqrt{\epsilon+1}\, e^{-\rho/2}\, \rho^{\gamma-1}(f_1+f_2) \qquad \dots \quad (18)$$

Writing $\Omega=\zeta/\sqrt{\epsilon^2-1}$, substituting (17) & (18) into (6) & (7) etc., we get finally

$$f_1'=\left[-\frac{(\gamma+i\Omega\epsilon)}{\rho}\right]f_1+\left[\left(j+\frac{1}{2}\right)-i\Omega\right]\frac{f_2}{\rho} \qquad \dots \quad (19)$$

$$f_2'=\left[1-\frac{(\gamma-i\Omega\epsilon)}{\rho}\right]f_1+\left[j+\frac{1}{2}+i\Omega\right]\frac{f_2}{\rho} \qquad \dots \quad (20)$$

where prime denotes differentiation with respect to ρ.

Differentiating (19) with respect to ρ, we get finally

$$f_1''+\left[\frac{(2\gamma+1)}{\rho}-1\right]f_1'+\left[-\frac{(\gamma+i\Omega\epsilon)}{\rho}+\frac{\{\gamma^2+\Omega^2\epsilon^2-(j+\frac{1}{2})^2-\Omega^2\}}{\rho^2}\right]f_1=0 \qquad \dots \quad (21)$$

We use

$$\gamma^2=\left(j+\frac{1}{2}\right)^2-\zeta^2 \qquad \dots \quad (22)$$

as determined from the condition of existence of solutions of type (17) & (18) as established for the bound muon case [Chatterjee]. We then get

$$f_1''+\left[\frac{(2\gamma+1)}{\rho}-1\right]f_1'+\left[-\frac{(\gamma+i\,\Omega\,\epsilon)}{\rho}\right]f_1=0 \qquad \dots \quad (23)$$

The solution of (23), regular at $r=0$ is

$$f_1=F(\gamma+i\Omega\epsilon,\ 2\gamma+1,\ \rho) \qquad \dots \quad (24)$$

From (19) we write

$$f_2=\frac{\rho}{\{j+\frac{1}{2}-i\,\Omega\}}\left[f_1'+\frac{(\gamma+i\,\Omega\,\epsilon)\,f_1)}{\rho}\right]$$

Using the differentiation properties of the Confluent function, we get finally

$$f_2=\frac{(\gamma+i\,\Omega\,\epsilon)}{(j+\frac{1}{2}-i\,\Omega)}\,[F(\gamma+i\,\Omega\,\epsilon+1,\ 2\gamma+1,\ \rho)] \qquad \dots \quad (25)$$

(iii) **The Phase Shift :** The phase shift δ_j is determined from the boundary conditions as is usual for cut-off potentials (Schiff 1955).

δ_j is defined by

$$\tan\delta_j=-c_j^2/c_j^1 \qquad \dots \quad (26)$$

To find c_j^2/c_j^1 from the boundary conditions, we equate the functions $\psi_{internal}$ and $\psi_{external}$ at $r=a_1$.

$$A_j \begin{pmatrix} f^{in}_{a_1} T^1_j \, y_{j+\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ -f^{in}_{a_1} T^2_j \, y_{j+\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \\ G^{in}_{a_1} T^3_j \, y_{j-\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ G^{in}_{a_1} T^4_j \, y_{j-\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} + A_{-j-1} \begin{pmatrix} g^{in}_{a_1} T^3_j \, y_{j-\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ g^{in}_{a_1} T^4_j \, y_{j-\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \\ F^{in}_{a_1} T^1_j \, y_{j+\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ -F^{in}_{a_1} T^2_j \, y_{j+\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix}$$

$$= \frac{C^1_j}{\sqrt{p_0 a_1}} \begin{pmatrix} J_{j+1}(p_0 a_1) \; T^1_j \; y_{j+\frac{1}{2},\, m-\frac{1}{2}} \; (\theta,\varphi) \\ -J_{j+1}(p_0 a_1) \; T^2_j \; y_{j+\frac{1}{2},\, m+\frac{1}{2}} \; (\theta,\varphi) \\ J_j(p_0 a_1) \dfrac{i\hbar cp}{W_1} \, T^3_j \; y_{j-\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ J_j(p_0 a_1) \dfrac{i\hbar cp}{W_1} \, T^4_j \; y_{j-\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix}$$

$$+ \frac{C^2_j}{\sqrt{p_0 a_1}} \begin{pmatrix} N_{j+1}(p_0 a_1) \; y_{j+\frac{1}{2},\, m-\frac{1}{2}} \; (\theta,\varphi) T^1_j \\ -N_{j+1}(p_0 a_1) \; y_{j+\frac{1}{2},\, m+\frac{1}{2}} \; (\theta,\varphi) T^2_j \\ \dfrac{i\hbar cp}{W_1} N_j(p_0 a_1) \; y_{j-\frac{1}{2},\, m-\frac{1}{2}} \; (\theta,\varphi) T^3_j \\ \dfrac{i\hbar cp}{W_1} N_j(p_0 a_1) \; y_{j-\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) T^4_j \end{pmatrix}$$

$$+ \frac{C^1_{-j-1}}{\sqrt{p_0 a_1}} \begin{pmatrix} J_{-j}(p_0 a_1) \, T^3_j \; y_{j-\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ J_{-j}(p_0 a_1) \, T^4_j \; y_{j-\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \\ \dfrac{i\hbar cp}{W_1} \; J_{-j-1}(p_0 a_1) \; T^1_j \; y_{j+\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ -\dfrac{i\hbar cp}{W_1} \; J_{-j-1}(p_0 a_1) \; T^2_j \; y_{j+\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix}$$

$$+ \frac{C^2_{-j-1}}{\sqrt{p_0 a_1}} \begin{pmatrix} N_{-j}(p_0 a_1) \, T^3_j \; y_{j-\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ N_{-j}(p_0 a_1) \, T^4_j \; y_{j-\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \\ \dfrac{i\hbar cp}{W_1} N_{-j-1}(p_0 a_1) \; T^1_j \; y_{j+\frac{1}{2},\, m-\frac{1}{2}}(\theta,\varphi) \\ -\dfrac{i\hbar cp}{W_1} N_{-j-1}(p_0 a_1) \; T^2_j \; y_{j+\frac{1}{2},\, m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \quad \dots \quad (27)$$

Equating coefficients of the same Spherical harmonics, we have

$$\sqrt{p_0 a_1}\, A_j\, f^{in}_{a_1} = C_j^1\, J_{j+1}(p_0 a_1) + C_j^2\, N_{j+1}(p_0 a_1) \qquad \cdots \quad (28)$$

$$\frac{W_1 \sqrt{p_0 a_1}}{i\hbar cp}\, A_j\, G^{in}_{a_1} = C_j^1\, J_j(p_0 a_1) + C_j^2\, N_j(p_0 a_1) \qquad \cdots \quad (29)$$

and two similar equations for A_{-j-1}, δ_{-j-1} derived from the set (8), (9)

$$\therefore \quad \sqrt{p_0 a_1}\; \frac{A_j}{C_j}\, f^{in}_{a_1} = J_{j+1}(p_0 a_1) + \frac{C_j^2}{C_j^1}\, N_{j+1}(p_0 a_1) \qquad \cdots \quad (30)$$

Substituting (30) into (29),

$$\frac{W_1}{i\hbar cp}\; \frac{G^{in}_{a_1}}{f^{in}_{a_1}}\, [J_{j+1}(p_0 a_1) + D_j N_{j+1}(p_0 a_1)]$$

$$= J_j\,(p_0 a_1) + D_j N_j(p_0 a_1) \qquad \cdots \quad (31)$$

where

$$D_j = -\tan \delta_j = C_j^2 / C_j^1 \qquad \cdots \quad (32)$$

$$\therefore\ \tan \delta_j = \frac{\left[J_j(p_0 a_1) - \dfrac{W_1}{i\hbar cp_0}\, \dfrac{G^{in}_{a_1}}{f^{in}_{a_1}}\; J_{j+1}(p_0 a_1) \right]}{\left[\dfrac{W_1}{i\hbar cp_0}\, \dfrac{G^{in}_{a_1}}{f^{in}_{a_1}}\; N_{j+1}(p_0 a_1) - N_j(p_0 a_1) \right]} \qquad \cdots \quad (33)$$

Similarly we find tan δ_{-j-1} from the set (8) and (9).

The Bessel functions can be written as

$$J^{(\rho)}_{l+\frac{1}{2}} = \left(\frac{2\rho}{\pi}\right)^{\frac{1}{2}} j_l^{(\rho)} \text{ and } N^{(\rho)}_{l+\frac{1}{2}} = \left(\frac{2\rho}{\pi}\right)^{\frac{1}{2}}\ \eta_l^{(\eta)}\ .$$

$$\therefore \quad \tan \delta_j = -\frac{\left[j_{j-\frac{1}{2}}(p_0 a_1) - \dfrac{W_1}{i\hbar\, cp_0}\, \dfrac{G^{in}_{a_1}}{f^{in}_{a_1}}\; j_{j+\frac{1}{2}}(p_0 a_1) \right]}{\left[\dfrac{W_1}{i\hbar\, cp_0}\, \dfrac{G^{in}_{a_1}}{f^{in}_{a_1}}\; \eta_{j+\frac{1}{2}}(p_0 a_1) - \eta_{j-\frac{1}{2}}(p_0 a_0) \right]} \qquad \cdots \quad (34)$$

Substituting the expression for $f^{in}_{a_1}$, $G^{in}_{a_1}$,

$$\tan \delta_j = \frac{\left[\sqrt{\epsilon-1}\,(f_1-f_2)_{a_1}\; j_{j-\frac{1}{2}}(p_0a_1) - \frac{W_1\sqrt{\epsilon+1}}{i\hbar cp_0}(f_1+f_2)_{a_1}\; j_{j+\frac{1}{2}}(p_0a_1) \right]}{\left[\frac{W_1}{i\hbar cp_0}\sqrt{\epsilon+1}\,(f_1+f_2)_{a_1}\; \eta_{j+\frac{1}{2}}(p_0a_1) - \sqrt{\epsilon-1}\;\eta_{j-\frac{1}{2}}(p_0a_1)\;(f_1-f_2)_{a_1} \right]} \quad \ldots \quad (35)$$

(iv) **The field free limit :** As the potential goes to zero,

$$ze^2 \to 0,\ \Omega = \frac{\zeta}{\sqrt{1-\epsilon^2}} \to 0,\ ze^2\left(\frac{1}{r}-\frac{1}{a_1}\right) \to 0\,.$$

$$f_2 \to F(a, c, 2D),\ \epsilon \to 1,\ \sqrt{1-\epsilon^2} \to 0,\ p \to 0,\ p_0 \to 0$$

and $F(a, c, 2D) \to 1$, since $2D \to 0$.

also $\sqrt{\epsilon-1} \to 0$ and $\frac{W_1\sqrt{\epsilon+1}}{i\hbar cp_0}(f_1+f_2)_{a_1}$

cancels from numerator and denominator. So

$$-\tan \delta_j \to \frac{j_{j+\frac{1}{2}}(p_0a_1)}{\eta_{j-\frac{1}{2}}(p_0a_1)} \to \frac{(p_0\,a_1)^{j+\frac{1}{2}}(-1)\,(p_0\,a_1)^{j+\frac{3}{2}}}{\{2(j+\frac{1}{2})+1\}\,!!\,\{2(j+\frac{1}{2})-1\}\,!!} \text{ for small } (p_0\,a_1).$$

as $\tan \delta_j \to 0$ as $p_0 \to 0$ and $\tan \delta_j$ decreases with increasing j.

(v) **The Scattering amplitude :** To find the scattering amplitude, we write the wave function in the form

$$\psi_j \xrightarrow[r\to\alpha]{} A_v\, e^{ikz} + B_j \frac{e^{ikr}}{r} f(\theta) \quad \ldots \quad (36)$$

to represent a plane wave and an outgoing wave as $r \to$ infinity. The plane wave constant A_v can be determined by expanding e^{ikz} in terms of spherical waves involving the free particle solutions for $(V(r))=0$, regular at $r=0$. (Rose 1961).

Our solutions (3) for the case $r \geqslant a_1$, should then tend to the form given by (36) as $s \to \alpha$

For S wave scattering, $j=1/2$, $l=0$

$$\psi_{v_2} = A_{v_2}\begin{pmatrix} f(r)\,T_j^1\,y_{1,\,m-\frac{1}{2}}(\theta,\varphi) \\ -f(r)\,T_j^2\,y_{1,\,m+\frac{1}{2}}(\theta,\varphi) \\ G(r)\,T_j^3\,y_{0,\,m-\frac{1}{2}}(\theta,\varphi) \\ G(r)\,T_j^4\,y_{0,\,m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} + A_{-\frac{1}{2}-1}\begin{pmatrix} g(r)\,T_j^3\,y_{0,\,m-\frac{1}{2}}(\theta,\varphi) \\ g(r)\,T_j^4\,y_{0,\,m+\frac{1}{2}}(\theta,\varphi) \\ F(r)\,T_j^1\,y_{1,\,m-\frac{1}{2}}(\theta,\varphi) \\ -F(r)\,T_j^2\,y_{1,\,m+\frac{1}{2}}(\theta,\varphi) \end{pmatrix} \quad \ldots \quad (37)$$

The asymptotic expansions of the radial functions $f(r)$, $G(r)$, $F(r)$, $G(r)$ are then substituted into (36) the plane wave is subtracted, and coefficients of ϵ^{-kr}/r are set equal to zero. This condition determines the remaining constant and the coefficient of e^{-ikr}/r gives $f(\theta)$ in terms of the phase shifts δ_j, δ_{-j-1}. The non-relativistic limit can be obtained by setting [Mott and Massey 1933]

$$\delta_j = \delta_{-j-1} \qquad \cdots \quad (38)$$

In writing the asymptotic expansions, it is convenient to write the external solutions in terms of the Hankel functions

$$h_j^{(1)}(p_0 r) \text{ and } h_j^{(2)}(p_0 r).$$

Then at $r \to \propto$, the 2 functions behave as $e^{ip_0 r}/r$ and $e^{-ip_0 r}/r$ respectively, so that the coefficient of $h_j^{(2)}(p_0 r)$ can straight away be put equal to zero at $r \to \propto$.

(vi) **Discussion :** While in most cases screening by atomic electrons is taken into account by additional form factors or the Hartree Fock technique, the use of a simplified model seems to enable us to visualise the screening effects directly. The abrupt cut-off nature of the Allis Morse Potential in particular can be used to explain any possible anomalous behaviour which other types of continuously varying screened potentials cannot be expected to do. Further, the presence of the screening parameter enables us to conveniently use it as a fitting parameter to interpret the experimental data.

References

Allis and Morse, (1931), *Zeits. f. Physik* **70**, 567.

Chatterjee L., Energy-Levels of a Muonic Atom. To Be Published in Annalen-der-Physik.

Hughes V. and Kinoshita T., Muon Physics, chap I, Vol. I, Academic Press, N. Y.-London.

Mott N. F. and Massey H. S. W., Theory of atomic collissions. Oxford (1933).

Mott N. F. and Sneddon I. N., Wave Mechanics and its applications. Dover Publications N. Y. (1948).

Rose M. E. (1961), Relativistic electron theory. John Wiley and Sons Inc.

Schiff L. I., (1955), Quantum Mechanics. McGraw hill Book Company.

JADAVPUR UNIVERSITY
DEPARTMENT OF PHYSICS
CALCUTTA-32

Bull. Cal. Math. Soc., 75, 153—160 (1983)

STEADY STATE SOLUTION OF SOME HEAT CONDUCTION PROBLEMS UNDER MIXED BOUNDARY CONDITIONS

RANJIT KUMAR CHAKRABORTY

(*Received 25 April 1979*)

Abstract : The existence of the solution of the two dimensional heat conduction equation under symmetric boundary condition when the rigion under consideration is semi-infinite strip or circular boundary has been discussed.

1. Introduction : In the previous paper (Chakraborty...) we have already discussed the solution of the dual trigonometric sine series under anti-symmetric boundary conditions. In this paper the solution of dual cosine series under symmetric boundary conditions has been discussed.

Solution of two dimensional heat conduction problem, under symmetric boundary condition depends on the solution of dual trigonometric cosine series of the type

$$\left.\begin{aligned}\lambda a_0+\sum_{n=1}^{\infty} n a_n \cos nx &= f_1(x),\ x\in(0,c)\\ \sum_{n=0}^{\infty} a_n \cos nx &= f_2(x),\ x\in(c,\pi)\end{aligned}\right\}\qquad(1.1)$$

where $f_1(x)$ and $f_2(x)$ are functions of x and λ is an arbitrary constant but a_n's are unknown constants to be determined.

The formal solution of (1.1) has been discussed by (1970) Tranter (1964), Srivastav (1964), Sneddon (1966), Noble and Whiteman (1970) have obtained the solution for $L^2(0,\pi)$. We first obtain formal solutions of (1.1 by complex variable technique, then we establish existance of the solution in $L(0,\pi)$.

Our object will be to determine an analytic function $u(x,y)+iv(x,y)$ of the complex variable $z=x+iy$ in the region $-\pi<x<\pi$ and $y>0$, such that $u(x,y)$ satisfy the following boundary conditions

(i) $\dfrac{\partial u}{\partial y}=-f_1(x)+\lambda a_0,\ y=0,\ x\in(-c,c)$

(ii) $u=f_2(x)$ on $y=0,\ |x|\in(c,\pi)$

(iii) $\dfrac{\partial u}{\partial x}=0$ on $x=\pm\pi$ and

(iv) $|u|<\infty$ as $y\to\infty$

where $f_1(x)$ and $f_2(x)$ symmetric function of x.

2. Consider the pair of dual series

$$\left.\begin{aligned} \sum_{n=1}^{\infty} n b_n \cos nx &= f_1(x) - \lambda(b_0+c_0), \; x\in(0,c)\\ \sum_{n=0}^{\infty} b_n \cos nx &= 0 \qquad\qquad , \; x\in(c,\pi)\end{aligned}\right\} \qquad (2.1)$$

and

$$\left.\begin{aligned} \sum_{n=1}^{\infty} n c_n \cos nx &= 0 \qquad\qquad , \; x\in(0,c)\\ \sum_{n=0}^{\infty} c_n \cos nx &= f_2(x) \qquad , \; x\in(c,\pi)\end{aligned}\right\} \qquad (2.2)$$

Therefore $a_n=b_n+c_n$ are the solution of (1.1) provided b_n and c_n satisfy (2.1) and (2.2) respectively.

To determine b_n, consider the bounded analytic function

$$\psi(z)=u(x,y)+iv(x,y)=\int_0^c \frac{\cos(z/2)\,f(t)}{\sqrt{(\cos z-\cos t)}}\,dt+ik \qquad (2.3)$$

By first of equation (2.1) and (2.3) we have

$$F(x)=\int_0^x \frac{f(t)\,dt}{\sqrt{(\cos t-\cos x)}}, \text{ where } F(x)=\int_0^x [f_1(t)-\lambda(b_0+c_0)]\,dt$$

The solution of the above Abel equation is

$$f(x)=\frac{2}{\pi}\int_0^x \frac{[f_1(t)-\lambda(b_0+c_0)]\cos t/2 \tan x/2}{\sqrt{(\cos t-\cos x)}}\,dt \qquad (2.4)$$

provided $f_1(x)$ is continuous. It is well known that [Erdelyi 1954],

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(\cos\theta)\cos(n+\tfrac{1}{2})x &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{(\cos x-\cos\theta)}}, \quad \theta>x\\ &=0 \qquad\qquad , \theta<x\end{aligned}$$

and

$$1+\sum_{n=1}^{\infty}[P_{n-1}(\cos\theta)+P_n(\cos\theta)]\cos nx=\frac{\sqrt{2}\cos(x/2)\,H(\theta-x)}{\sqrt{(\cos x-\cos\theta)}}. \qquad (2.5)$$

where $H(t)$ is the Heaviside unit function. Since $\{\cos nx/n=0, 1,\ldots)$ is complete in a set S of all even function defined in $L^2(-\pi,\pi)$, hence (2.3), (2.4), (2.5) and

$$u(x,0)=\sum_{n=0}^{\infty} b_n \cos nx$$

$$\rightarrow b_n=-\lambda(b_0+c_0)\int_a^c [P_n(\cos\theta)+P_{n-1}(\cos\theta)]\tan\theta/2\,d\theta+$$

$$\left.\begin{aligned} &+\frac{\sqrt{2}}{\pi}\int_0^c [P_n(\cos\theta)+P_{n-1}(\cos\theta)]\tan\theta/2\,d\theta\int_0^\theta \frac{f_1(t)\cos t/2}{\sqrt{(\cos t-\cos\theta)}}\,dt \quad . \\ &\qquad\qquad\qquad\qquad n=1,2,3,\ldots \\ &b_0=-2\lambda\,(b_0+c_0)\log\sec(c/2)+\frac{\sqrt{2}}{\pi}\int_0^c \tan\theta/2\,d\theta\int_0^\theta\frac{f_1(t)\,dt}{\sqrt{(\cos t-\cos\theta)}}\,. \end{aligned}\right\} \tag{2.6}$$

To determine c_n from (2.2) we have to consider the analytic function

$$\phi(z)=\int_c^\pi \frac{g(t)\cos(z/2)\,dt}{\sqrt{(\cos z-\cos t)}}\,. \tag{2.7}$$

From second of equation (2.2) and (2.7) we get

$$\int_x^\pi \frac{g(t)\,dt}{\sqrt{(\cos x-\cos t)}}=\frac{f_2(x)}{\cos x/2}\,,\qquad c<x<t<\pi.$$

The solution of the above equation is

$$g(x)=-\frac{2}{\pi}\frac{d}{dx}\int_x^\pi \frac{f_2(t)\sin(t/2)}{\sqrt{(\cos x-\cos t)}}\,dt.$$

Therefore Re $\varphi(z)$ on $y=0$ is equal to

$$\left.\begin{aligned} &-\frac{2}{\pi}\int_c^\pi \frac{\cos x/2\,d\theta}{\sqrt{(\cos x-\cos\theta)}}\frac{d}{d\theta}\int_\theta^\pi \frac{f_2(t)\sin t/2}{\sqrt{(\cos\theta-\cos t)}}\,dt,\quad 0<x<c \\ &\text{and} \\ &-\frac{2}{\pi}\int_x^\pi \frac{\cos x/2\,d\theta}{\sqrt{(\cos x-\cos\theta)}}\frac{d}{d\theta}\int_\theta^\pi \frac{f_2(t)\sin t/2}{\sqrt{(\cos\theta-\cos t)}}\,dt,\quad c<x<\pi \end{aligned}\right\} \tag{2.8}$$

It follows from (2.5), (2.7) and (2.8)

$$\left.\begin{aligned} &c_n=-\frac{\sqrt{2}}{\pi}\int_c^\pi [P_n(\cos\theta)+P_{n+1}(\cos\theta)]\,d\theta\,\frac{d}{d\theta}\int_\theta^\pi \frac{f_2(t)\sin t/2\,dt}{\sqrt{(\cos\theta-\cos t)}} \\ &\qquad\qquad\qquad\qquad n=1,2,3,\cdots \\ &c_0=-\frac{\sqrt{2}}{\pi}\int_c^\pi d\theta\,\frac{d}{d\theta}\int_\theta^\pi \frac{f_2(t)\sin t/2\,dt}{\sqrt{(\cos\theta-\cos t)}}\,. \end{aligned}\right\} \tag{2.9}$$

From (2.6) and (2.9) we have to following expression of a_n and a_0

$$\begin{aligned} a_n=&-\lambda(b_0+c_0)\int_0^c [P_n(\cos\theta)+P_{n-1}(\cos\theta)]\tan\theta/2\,d\theta \\ &+\frac{\sqrt{2}}{\pi}\int_0^c [P_n(\cos\theta)+P_{n-1}(\cos\theta)]\tan\theta/2\,d\theta\int_0^\theta \frac{f_1(t)\cos t/2}{\sqrt{(\cos t-\cos\theta)}}\,dt \\ &-\frac{\sqrt{2}}{\pi}\int_c^\pi [P_n(\cos\theta)+P_{n-1}(\cos\theta)]\,d\theta\,\frac{d}{d\theta}\int_\theta^\pi \frac{f_2(t)\sin t/2}{\sqrt{(\cos\theta-\cos t)}}\,dt, \\ &\qquad\qquad\qquad\qquad n=1,2,3,\ldots\ldots \end{aligned}$$

and

$$a_0 = \frac{1}{1+2\lambda \log \sec (c/2)}\left[\frac{\sqrt{2}}{\pi}\int_0^c \tan \theta/2 \, d\theta \int_0^\theta \frac{f_1(t)\, dt}{\sqrt{(\cos \theta - \cos t)}}\right.$$

$$\left. -\frac{\sqrt{2}}{\pi}\int_c^\pi d\theta \, \frac{d}{d\theta}\int_\theta^\pi \frac{f_2(t) \sin t/2 \, dt}{\sqrt{(\cos \theta - \cos t)}}\right].$$

3. To obtain the solutions of (1.1) we have assumed that the term by term differentiation and integration of an infinite series are permissible. But in general this is not possible. Hence we shall formulate more general conditions for the existence of the solution of the system (1.1). It fact we have the following theorems,

Theorem 1. *Let $f_1(x)$ be a function defined on $(0, c)$ and satisfies Direchlet's conditions in the same interval, then*

$$\sum_{n=1}^{\infty} nb_n \cos nx = \tfrac{1}{2}\left[f_1 (x+0)+f_1 (x-0)-2\lambda (b_0+c_0) \right], \quad x\epsilon\ (0, c) \tag{3.1}$$

$$\sum_{n=0}^{\infty} b_n \cos nx = 0 \qquad , \quad x\epsilon\ (c, \pi) \tag{3.2}$$

where b_n and b_0 given in (2 6).

Theorem 2. *Let $f_2(x)$ be a function of x defined on (c, π) and $f_2'(x)$ exists almost every where in (c, π). Moreover $f_2(x)$ and $f_2'(x)$ satisfy Direchlet's conditions in the same interval, then*

$$\sum_{n=1}^{\infty} nc_n \cos nx = 0 \qquad , \quad x\epsilon\ (0, c) \tag{3.3}$$

$$\sum_{n=0}^{\infty} c_n \cos nx = \tfrac{1}{2}\left[f_2 (x+0)+f_2 (x-0) \right] \qquad , \quad x\epsilon\ (c, \pi) \tag{3.4}$$

where c and c_0 given in (2.9).

To prove the Theorem 1, consider (3.3). From integral representation of $P_n (\cos \theta)$ it follows that

$$b_n = \frac{4\lambda (b_0+c_0)}{n\pi}\int_c^\pi \cos n\phi \, \frac{d}{d\phi}\left[arc \cos \left(\frac{\cos \phi/2}{\cos c/2}\right)\right] d\phi$$

$$+ \frac{2}{n\pi}\int_0^c f_1(t) \cos nt \, dt - \frac{4}{n\pi^2}\int_c^\pi \cos n\phi \, d\phi \int_0^\theta f_2(t) \, \frac{d}{d\phi}$$

$$\left\{ arc \tan \left(\frac{\cos t/2}{\cos \phi/2}\sqrt{\left(\frac{\cos c - \cos \phi}{\cos t - \cos c}\right)}\right)\right\} dt.$$

Therefore from the above expression of b_n, it is clear tha

$$S_m(x) = \sum_{n=1}^{m} nb_n \cos nx$$

$$= \frac{2\lambda\,(b_0+c_0)}{\pi}\int_c^{\pi}\left[\frac{\sin\,(2m+1/2)\,(\phi+x)}{2\sin\,(\phi+x/2)}+\frac{\sin\,(2m+1/2)\,(\phi-x)}{2\sin\,(\phi-x/2)}-1\right]$$

$$\times\frac{d}{d\phi}\left[arc\cos\left(\frac{\cos\phi/2}{\cos c/2}\right)\right]d\phi$$

$$+\frac{1}{\pi}\int_0^{c}\left[\frac{\sin\,(2m+1/2)\,(t+x)}{2\sin\,(t+x/2)}+\frac{\sin\,(2m+1/2)\,(t-x)}{2\sin\,(t-x/2)}-1\right]f_1(t)\,dt$$

$$-\frac{2}{\pi^2}\int_c^{\pi}\left[\frac{\sin\,(2m+1/2)\,(\phi+x)}{2\sin\,(\phi+x/2)}+\frac{\sin\,(2m+1/2)\,(\phi-x)}{2\sin\,(\phi-x/2)}-1\right]d\phi$$

$$\times\int_0^{c} f_1(t)\;\frac{d}{d\phi}\;arc\tan\left(\frac{\cos t/2}{\cos c/2}\sqrt{\left(\frac{\cos c-\cos\phi}{\cos t-\cos c}\right)}\right)dt$$

Hence for all $x\epsilon$ (0, c)

$$\lim_{m\to\infty} S_m(x) = \tfrac{1}{2}\,[f_1\,(x+0)+f_1\,(x-0)\,]-\lambda\,(b_0+c_0)$$

Again from (3.3) and Mehler-Direchlet formula,

$$\forall\ x\geq 1,\ b_n = \frac{4}{\pi^2}\int_0^{c}\cos n\phi\cos\phi/2\,d\phi\int_{\phi}^{c}\frac{\tan\theta/2\,d\theta}{\sqrt{(\cos\phi-\cos\theta)}}\int_0^{\theta}\frac{f_1(t)\cos t/2\,dt}{\sqrt{(\cos\phi-\cos\theta)}}$$

$$-\frac{2\sqrt{2}\,\lambda\,(b_0+c_0)}{\pi}\int_0^{c}\cos n\phi\cos\phi/2\,d\phi\int_{\phi}^{c}\frac{\tan\theta/2\,d\theta}{\sqrt{(\cos\phi-\cos\theta)}}\,.$$

Since,

$$1 = P_0\,(\cos\theta) = \frac{\sqrt{2}}{\pi}\int_0^{\theta}\frac{\cos\phi/2\;d\phi}{\sqrt{(\cos\phi-\cos\theta)}}\,.$$

Therefore from (3.3)

$$b_0 = -\frac{\sqrt{2}\,\lambda\,(b_0+c_0)}{\pi}\int_0^{c}\cos\phi/2\,d\phi\int_{\phi}^{c}\frac{\tan\theta/2\,d\theta}{\sqrt{(\cos\phi-\cos\theta)}}$$

$$+\frac{2}{\pi^2}\int_0^{c}\cos\phi/2\,d\phi\int_{\phi}^{c}\frac{\tan\theta/2\,d\theta}{\sqrt{(\cos\phi-\cos\theta)}}\int_0^{\theta}\frac{f_1(t)\cos t/2\,dt}{\sqrt{(\cos t-\cos\theta)}}\,.$$

Hence

$$s_m(x) = \sum_{n=0}^{\infty} b_n\cos nx$$

$$= \frac{1}{\pi^2}\int_0^{c}\left[\frac{\sin\,(2m+1/2)\,(\phi+x)}{\sin\,(\phi+x/2)}+\frac{\sin\,(2m+1/2)\,(\phi-x)}{\sin\,(\phi-x/2)}\right]\cos\phi/2\,d\phi$$

$$\times\int_{\phi}^{c}\frac{\tan\theta/2\,d\theta}{\sqrt{(\cos\phi-\cos\theta)}}\int_0^{\theta}\frac{f_1(t)\cos t/2\,dt}{\sqrt{(\cos t-\cos\theta)}}$$

$$-\frac{\surd 2\lambda\,(b_0+c_0)}{2\pi}\int_0^c \left[\frac{\sin\,(2m+1/2)\,(\phi+x)}{\sin\,(\phi+x/2)}+\frac{\sin\,(2m+1/2)\,(\phi-x)}{\sin\,(\phi-x/2)}\right]\cos\phi/2\,d\phi$$

$$\times\int_\phi^c \frac{\tan\theta/2\,d\theta}{\surd(\cos\phi-\cos\theta)} \to 0 \quad \text{as} \quad m\to\infty.$$

we shall now prove Theorem 2. Since

$$P_n\,(\cos\theta) = \int_0^\theta \frac{\sqrt{2}}{\pi}\;\frac{(\cos\,(n+\frac{1}{2})\,\phi\,d\phi}{\sqrt{(\cos\phi-\cos\theta)}}$$

we have for $n \geqslant 1$

$$c_n = -\frac{4}{\pi^2}\left[\int_0^c \cos n\phi\,d\phi\int_c^\pi f_2{}'(t)\;arc\tan\left(\frac{\cos\phi/2}{\cos t/2}\sqrt{\left(\frac{\cos c-\cos t}{\cos\phi-\cos c}\right)}\right)dt\right.$$

$$\left.+\frac{\pi}{2}\int_c^\pi \cos n\phi\,d\phi\int_\phi^\pi f_2{}'(t)\,dt\right]$$

Since, $$\sum_{n=0}^{\infty} c_n\cos nx = -\frac{1}{\pi^2}\int_0^c\left[\frac{\sin\,(2m+1/2)\,(\phi-x)}{\sin\,(\phi-x/2)}+\frac{\sin\,(2m+1/2)\,(\phi+x)}{\sin\,(\phi+x/2)}\right]d\phi$$

$$\times\int_c^\pi f_2{}'(t)\;arc\tan\left(\frac{\cos\phi/2}{\cos t/2}\sqrt{\left(\frac{\cos c-\cos t}{\cos\phi-\cos c}\right)}\right)dt$$

$$-\frac{1}{2\pi}\int_c^\pi\left[\frac{\sin\,(2m+1/2)\,(\phi-x)}{\sin\,(\phi-x/2)}+\frac{\sin\,(2m+1/2)\,(\phi+x)}{\sin\,(\phi+x/2)}\right][f_2(\pi-0)-f_2(\phi)]\,d\phi.$$

Therefore $\forall\, x\epsilon(c,\pi).\ \lim\limits_{m\to\infty}\ \sum\limits_{n=0}^{m} c_n\cos nx = \frac{1}{2}\,[f_2\,(x+0)+f_2\,(x-0)]$

i.e. we have the result (3.5). To obtain (3.4), consider

$$P_n\,(\cos\theta) = \frac{\surd 2}{\pi}\int_\theta^\pi \frac{\sin\,(n+\frac{1}{2})\,\phi\,d\phi}{\surd(\cos\theta-\cos\phi)}.$$

Since

$$c_n = -\frac{4}{n\pi^2}\int_c^\pi \cos n\phi\,d\phi\left[\int_c^\phi f_2{}'(t)\,\frac{d}{d\phi}\,arc\tanh\left(\frac{\cos\phi/2}{\cos t/2}\Big/\left(\frac{\cos c-\cos t}{\cos c-\cos\phi}\right)\right)dt\right.$$

$$\left.+\int_\phi^\pi f_2{}'(t)\,\frac{d}{d\phi}\;arc\tanh\left(\frac{\cos t/2}{\cos\phi/2}\Big/\left(\frac{\cos c-\cos\phi}{\cos c-\cos t}\right)\right)dt\right].$$

Therefore $$\sum_{n=1}^{m} nc_n\cos nx = -\frac{2}{\pi^2}\sum_{n=1}^{m}\int_c^\pi[\cos n\,(\phi+x)+\cos n\,(\phi-x)]\,d\phi$$

$$\times\left[\int_c^\phi f_2{}'(t)\,\frac{d}{d\phi}\;arc\tanh\left(\frac{\cos\phi/2}{\cos t/2}\Big/\left(\frac{\cos c-\cos t}{\cos c-\cos\phi}\right)\right)dt\right.$$

$$\left.+\int_\phi^\pi f_2{}'(t)\,\frac{d}{d\phi}\,arc\tanh\left(\frac{\cos t/2}{\cos\phi/2}\sqrt{\left(\frac{\cos c-\cos\phi}{\cos c-\cos t}\right)}\right)dt\right].$$

From the identity

$$\tfrac{1}{2}+\cos x+ \quad \ldots \quad \ldots \quad +\cos nx = \frac{\sin (2n+1/2)\, x}{2 \sin x/2},$$

and from

$$\lim_{n\to\infty} \int_a^b \frac{\sin nx}{\sin x} f(x)\, dx = 0, \qquad (b > a > 0)$$

it can be easily shown that for all $x \in (0, c)$

$$\lim_{m\to\infty} \sum_{n=1}^{m} nc_n \cos nx = 0,$$

Hence we have the theorem 2.

4. For numerical evaluation if $f_1(x) = x - x^2 + .25x^3$ in $0 < x < 1$, $f_2(x) = 1+2x-.25x^2$ in $1 < x < \pi$. The computed value of $b_0 = 0.0639263$, $c_0 = -1.0491488$ and $\lambda = -0.081519$ obtained from the relation

$$\int_0^1 \left[\frac{\partial u}{\partial y}\right]_{y=0} dx + \int_1^\pi \left[\frac{\partial u}{\partial y}\right]_{y=0} dx = 0$$

After performing the numerical integration of equation (3.3) we have the set of values of b_n. Taking the first 30 terms of b_n we calculate the nature of the unknown function in the other intervals

(i) $\sum b_n \cos nx$ in $0 < x < 1$

(ii) $\sum nb_n \cos nx$ in $1 < x < \pi$

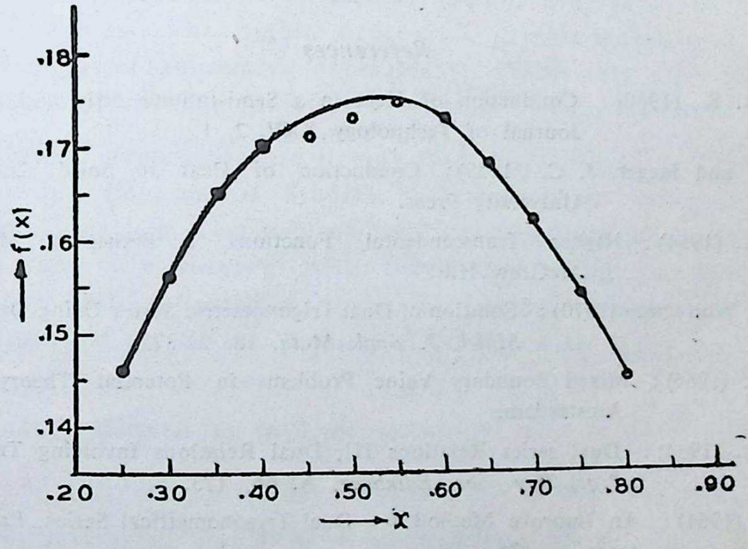


Fig. 1 – Curve for temperature distribution

and the corresponding nature of the curve for temperature distribution and normal flux given in Fig. 1. and Fig, 2.

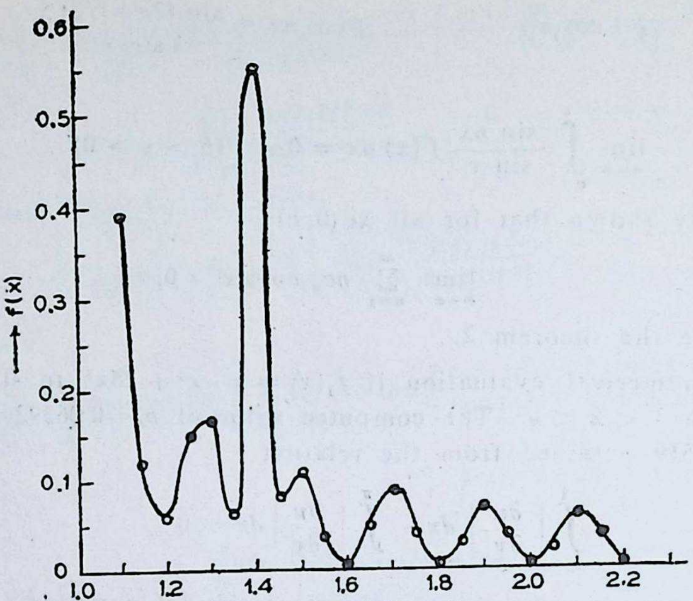


Fig. 2.– curve for normal flux

Acknowledgement : I take this opportunity of thanking Dr. P. K. Chaudhuri, for suggesting this investigation and for guidance at various stages of the work.

References

Chakraborty, R. K. (1980) : Conduction of Heat in a Semi-Infinite Strip and Circular Boundary, Journal of Technology XXV, 2, 1.

Carslaw, H. S. and Jaeger, J. C. (1959) : Conduction of Heat in Solid, 2nd. Edition, Oxford University Press.

Erdelyi et al A. (1954) : Higher Transcendental Functions, 1, Batheman Manuscript Project, McGraw Hill.

Noble, B, and Whiteman (1970) : Solution of Dual Trigonometric Series Using Orthogonal Relations. *SIMA, J. Appl. Math.* 18. 2. 372.

Sneddon, I. N. (1966) : Mixed Boundary Vaiue Problems in Potential Theory, North Holland, Amsterdam.

Srivastav, R. P. (1964) : Dual series Relations III, Dual Relatlons Involving Trigonometric Series, *Proc. Roy. Soc. Edinburg*, A, 66, 173.

Tranter, C. J. (1964) : An Improve Method for Dual Trigonometrical Series, *Proc. Glasgow, Math. Assoc.* 6. 136.

DEPARTMENT OF MATHEMATICS,
BENGAL ENGINEERING COLLEGE,
HOWRAH-711103, INDIA.

Bull. Cal. Math. Soc., 75 161 - 168, (1983)

ON AXIOMATIC CHARACTERISATION OF n-TH ORDER INFORMATION IMPROVEMENT

DIPA ROY (Nèè BOSE)

(*Received 11 July, 1979*)

Abstract: The author, in an earlier paper, introduced a quantity as a generalisation of Theil's Information Improvement and named it as Second order Information Improvement. In this paper as a further generalisation, the concept of n-th order Information Improvement is introduced. Then an axiomatic characterisation of an information theoretic quantity associated with n+2 probability distributions each having the same number of elements has been given. This quantity, under suitable conditions gives rise to n-th order Information Improvement.

1. Introduction: Theil's (1967) concept of "Information Improvement" (I.I.) found wide applications in economics and behavioral sciences. Roy (1978) went a step further towards generalisation by introducing a quantity called second order I.I.

The study of social sciences requires repeated statistical experimentations, sometimes at regular intervals. The results of these experiments are then analysed to infer the associated probability distributions which are likely to change with successive experimentations. The higher order I. I. gives a measure to the amount of information gained after successive experiments. With this idea in mind the concept of Information Improvement of *n*-th order has been developed.

Being a concept of general nature it may find applications in Communication System, Biomathematics, Economical Studies, particularly in cases that involve repeated experimentations (e.g. Census) calling for handling of huge amount of data and that too randomly varying with time. Depending on the results of experimentations, the data collected will go on varying. From these we can get more and more information about population that will help us in correct prediction.

n-th order I. I. is defined as follows:

Let X be a random variable taking m values $x_1, x_2, \dots, x_m$. Let $P=(p_1, p_2, \dots, p_m)$, $p_j \geq 0, \sum_{j=1}^{m} p_j=1$ be a probability distribution of m events on the basis of an experiment E. Suppose that the decomposition P is first predicted to be $Q=(q_1, \dots, q_m)$, $q_j \geq 0, \sum_{j=1}^{m} q_j=1$. Let, at an intermediate stage the prediction Q be first vised to $R^{(1)} = (r_1^{(1)}, r_2^{(1)}, \ldots, r_m^{(1)})$, $r_j^{(1)} \geqslant 0$, $\sum_{j=1}^{m} r_j^{(1)} = 1$. Then the revision $R^{(\ )}$ is revised to $R^{(2)} = (r_1^{(2)}, r_2^{(2)}, \ldots, r_m^{(2)})$, $r_j^{(2)} \geq 0$, $\sum_{j=1}^{m} r_j^{(2)} = 1$, $R^{(2)}$ to $R^{(3)} = (r_1^{(3)}, r_2^{(3)}, \ldots, r_m^{(3)})$, $r_j^{(3)} \geqslant 0$, $\sum_{j=1}^{m} r_j^{(3)} = 1$ and so on finally $R^{(n-1)}$ to $R^{(n)} = (r_1^{(n)}, r_2^{(n)}, \ldots, r_m^{(n)})$, $r_j^{(n)} \geq 0$, $\sum_{j=1}^{n} r_j^{(n)} = 1$.

Theil's I. I. is given by

$$I = I(P\,;Q\,;R^{(1)}) = \sum_{j=1}^{m} p_j \log \frac{r_j^{(1)}}{q_j}$$

This can be positive, zero or negative.

The original I. I. is $I(P\,;Q\,;R^{(1)})$ (after 1st revision), and the new value is $I(P\,;R^{(1)}\,;R^{(2)})$ (after 2nd revision). In an earlier paper of Roy (1978), second order I. I. is defined as the forward difference of these two I. I.'s (i.e. I.I. after 2nd revision minus I.I. after 1st revision)

$$\begin{aligned} \textit{i.e.}\quad I^2 &= I^2(P\,;Q\,;R^{(1)}\,;R^{(2)}) \\ &= I(P\,;R^{(1)}\,;R^{(2)}) - I(P\,;Q\,;R^{(1)}) \\ &= \sum_{j=1}^{m} p_j \log (r_j^{(2)}/r_j^{(1)}) - \sum_{j=1}^{m} p_j \log (r_j^{(1)}/q_j) \\ &= \sum_{j=1}^{m} p_j \log \frac{r_j^{(2)} q_j}{(r_j^{(1)})^2} \\ &= \sum_{j=1}^{m} p_j (\log r_j^{(2)} - 2 \log r_j^{(1)} + \log q_j), \end{aligned}$$

assuming that whenever a q_j or $r_j^{(1)}$ or $r_j^{(2)}$ is zero, corresponding $p_j = 0$ and $0 \log 0 = 0$.

Second order I. I. will be positive, zero or negative according as $I(P\,;R^{(1)}\,;R^{(2)}) \gtreqless I(P\,;Q\,;R^{(1)})$.

In like manner, the third order I. I. is the forward difference of two second order I. I.'s viz. $I^2(P\,;Q\,;R^{(1)}\,;R^{(2)})$ and $I^2(P\,;R^{(1)}\,;R^{(2)}\,;R^{(3)})$ (i.e. I^2 after 3rd revision minus I^2 after 2nd revision), so that third order I.I. is

$$\begin{aligned} I^3 &= I^3(P\,;Q\,;R^1\,;R^2\,;R^3) \\ &= I^2(P\,;R^1\,;R^2\,;R^3) - I^2(P\,;Q\,;R^1\,;R^2) \\ &= \sum_{j=1}^{m} p_j \log \frac{r_j^3 (r_j^1)^3}{(r_j^2)^3 q_j} \\ &= \sum_{j=1}^{m} p_j ({}^3C_3 \log r_j^{(3)} - {}^3C_2 \log r_j^{(2)} + {}^3C_1 \log r_j^{(1)} - {}^3C_0 \log q_j) \end{aligned}$$

assuming that whenever a q_j or $r_j^{(1)}$ or $r_j^{(2)}$ or $r_j^{(3)}$ is zero, corresponding $p_j = 0$ and $0 \log 0 = 0$.

It can be easily proved by mathematical induction that for a positive integer n, I. I. of nth order is the forward difference of two $(n-1)$ th order I. I.'s viz.

$$I^{n-1}(P ; Q, R^{(1)} ; \cdots ; R^{(n-1)}) \quad \text{and} \quad I^{n-1}(P ; R^{(1)} ; R^{(2)} ; \ldots ; R^{(n)})$$

(i.e. I^{n-1} after n-th revisions minus I^{n-1} after $n-1$th revisions) i.e. n-th ordet I.I. is

$$I^n = I^n(P ; Q ; R^{(1)} ; \ldots , R^{(n)})$$

$$= I^{n-1}(P ; R^{(1)} ; \cdots , R^{(n)}) - I^{n-1}(P ; Q ; R^{(1)} ; \cdots , R^{(n-1)})$$

$$= \sum_{j=1}^{m} p_j \left({}^nC_n \log r_j^{(n)} - {}^nC_{n-1} \log r_j^{(n-1)} + {}^nC_{n-2} \log r_j^{(n-2)} \right.$$

$$\left. - \ldots + (-1)^{n-1}\, {}^nC_1 \log r_j^{(1)} + (-1)^n\, {}^nC_0 \log q_j \right)$$

assuming that whenever a q_j or $r_j^{(1)}$ or $r_j^{(2)}$ or $\cdots$ or $r_j^{(n)}$ is zero, corresponding $p_j=0$ and $0 \log 0=0$. ce similar assumptions have been made in case of I.I.'s of all orders ($<n$) the possibility of any of the I^m for $m \leq n-1$ being undefined is obviously excluded. Also since the values of I^n is deduced from the first principle of induction, therefore all I^m for $m=2, 3, \cdots, n$ exist.

The value of $I^n\,(n \geqslant 3)$ will be positive, zero or negative according as $I^{n-1}(P ; R^{(1)} ; \ldots , R^{(n)}) \gtreqless I^{n-1}(P ; Q ; R^{(1)} ; \ldots , R^{(n-1)})$. The value of I.I. of any otder indicates "improvement" in the literal sense when positive, deterioration when negative and stationarity when zero.

Remark: Throughout this paper the logarithm is taken to the base 2, unless otherwise mentioned.

In this paper, an axiomatic characterisation of information theoretic quantity associated with $n+2$ probability distributions has been given. This quantity under suitable conditions leads to nth order I. I.

2. Characterisation Theorem: For each positive integer m, let s_m be an information theoretic measure which is a continuous function of $(n+2)\,m$ variables $p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_m ; r_1^{(1)}, r_2^{(1)}, \ldots , r_m^{(1)} ; \ldots ; r_1^{(n)}, r_2^{(n)}, \ldots , r_m^{(n)}$ where $p_j, q_j, r_j^{(1)}, \cdots , r_j^{(n)}$ are $n+2$ probability distributions on the subspace of $R^{(n+2)m}$ defined by the relations $p_j \geqslant 0$, $q_j \geqslant 0$, $r_j^{(1)} \geqslant 0, \ldots , r_j^{(n)} \geqslant 0$ and

$$\sum_{=1}^{m} \quad = 1, \qquad \sum_{j=1}^{m} q_j = 1, \qquad \sum_{j=1}^{m} r_j = 1, \quad \ldots , \quad \sum_{j=1}^{m} r_j^{(n)} = 1.$$

Let the function s_m have the following properties :

(*i*)
$$s_m \begin{bmatrix} p_1, & \dots, & p_m \\ q_1, & \dots, & q_m \\ r_1^{(1)} & \dots, & r_m^{(1)} \\ & \dots & \dots \\ r_1^{(n)} & \dots, & r_m^{(n)} \end{bmatrix} = s_m \begin{bmatrix} p_a^{(1)}, & \dots, & p_a^{(m)} \\ q_a^{(1)}, & \dots, & q_a^{(m)} \\ r^{(1)}_{a(1)} & \dots, & r^{(1)}_{a(m)} \\ \dots & & \dots \\ r^{(n)}_{a(1)}, & \dots, & r^{(n)}_{a(m)} \end{bmatrix}$$

where $a(1), \dots, a(m)$ is any permutation of $(1, 2, \dots, m)$.

(*ii*)
$$s_m \begin{bmatrix} p_1 & \dots, & p_{m-1}, & 0 \\ q_1 & \dots, & q_{m-1}, & q_m \\ r_1^{(1)} & \dots, & r_{m-1}^{(1)}, & r_m^{(1)} \\ \\ r_1^{(n)} & \dots, & r_{m-1}^{(n)}, & r_m^{(n)} \end{bmatrix} = s_{m-1} \begin{bmatrix} p_1, & \dots, & p_{m-1} \\ q_1, & \dots, & q_{m-1} \\ r_1^{(1)} & \dots, & r_{m-1}^{(1)} \\ \\ r_1^{(n)}, & \dots, & r_{m-1}^{(n)} \end{bmatrix}$$

(*iii*) If

$$\sum_{i=1}^{l} \sum_{j=1}^{m} p_{ij} = 1, \quad p_{ij} \geqslant 0, \quad p_j = \sum_{i=1}^{l} p_{ij} > 0,$$

$$\sum_{i=1}^{l} \sum_{j=1}^{m} q_{ij} = 1, \quad q_{ij} \geqslant 0, \quad q_j = \sum_{i=1}^{l} q_{ij} > 0,$$

$$\sum_{i=1}^{l} \sum_{j=1}^{m} r^{(1)}_{ij} = 1, \quad r^{(1)}_{ij} \geq 0, \quad r^{(1)}_j = \sum_{i=1}^{l} r^{(1)}_{ij} > 0,$$

$$\sum_{i=1}^{l} \sum_{j=1}^{m} r^{(n)}_{ij} = 1, \quad r^{(n)}_{ij} \geq 0, \quad r^{(n)}_j = \sum_{i=1}^{l} r^{(n)}_{ij} > 0,$$

then

$$s_{lm} \begin{bmatrix} p_{11}, & \dots, & p_{lm} \\ q_{11}, & \dots, & q_{lm} \\ r^{(1)}_{11}, & \dots, & r^{(1)}_{lm} \\ \\ r^{(n)}_{11}, & \dots, & r^{(n)}_{lm} \end{bmatrix} = s_m \begin{bmatrix} p_1, & \dots, & p_m \\ q_1, & \dots, & q_m \\ r_1^{(1)}, & \dots & r^{(1)} \\ \\ r_1^{(n)}, & \dots, & r_m^{(n)} \end{bmatrix}$$

$$+ \sum_{j=1}^{m} p_j s_l \begin{bmatrix} (p_{1j} / p_j), & \dots, & (p_{lj} / p_j) \\ (q_{1j} / q_j), & \dots, & (q_{lj} / q_j) \\ (r^{(1)}_{1j} / r^{(1)}_j), & \dots, & (r^{(1)}_{lj} / r^{(1)}_j) \\ \\ (r^{(n)}_{1j} / r^{(n)}_j), & \dots, & (r^{(n)}_{lj} / r^{(n)}_j) \end{bmatrix}$$

where jth term in the sum is understood to be zero if $p_j=0$.

(iv) $$s_1(1\,;\,1\,;\,1\,;\,\dots\,;\,r^{(n)}) = s_m\begin{bmatrix} 1/m, & \dots, & 1/m \\ 1/m, & \dots, & 1/m \\ 1/m, & \dots, & 1/m \\ & & \\ r^{(n)}/m, & \dots, & r^{(n)}/m \end{bmatrix}$$

Then

$$s_m\begin{bmatrix} p_1, & \dots, & p_m \\ q_1, & \dots, & q_m \\ r_1^{(1)}, & \dots, & r_m^{(1)} \\ & & \\ r_1^{(n)}, & \dots, & r_m^{(n)} \end{bmatrix} = A\sum_{j=1}^{m} p_j \log p_f + B\sum_{j=1}^{m} p_j \log q_j$$

$$+D_1\sum_{j=1}^{m} p_j \log r_j^{(1)} + \dots \dots + D_n\sum_{j=1}^{m} p_j \log r_j^{(n)} \tag{2.1}$$

where $A,\ B,\ D_1,\ D_2,\ \dots,\ D_n$ are arbitrary constants.

Proof: From (iii) and (iv), if $a>0$ and $b>0$

$$s_1(1\,;\,1\,;\,1\,;\,\cdots\,;\,1\,;\,ab) = s_1(1\,;\,1\,;\,1\,;\,\dots\,;\,1\,;\,a) + s_1(1\,;\,1\,;\,1\,;\,\dots\,;\,1\,;\,b)$$

This is a Cauchy equation (Aczel 1966).

$$f(xy) = f(x)+f(y)$$

for $$f(x) = s_1(1,\ 1,\ 1,\ \dots,\ 1,\ x)$$

with the solution

$$s_1(1,\ 1,\ 1,\ \dots,\ 1,\ r_j^{(n)}) = D_n \log r_j^{(n)},$$

if $s_1(1,\ 1,\ 1,\ \dots,\ 1,\ r_j^{(n)})$ is continuous.

Similarly we obtain

$$s_1(1,\ 1,\ \dots\ 1,\ r_j^{(k)},\ 1.\ \dots,\ 1) = D_k \log r_j^{(k)}$$

and $$s_1(1,\ q_j,\ 1,\ \dots,\ 1) = B\log q_j$$

Again from (iii) and (iv)

$$\begin{aligned} s_1(1,\ q_j,\ r_j^{(1)},\ 1,\ \dots,\ 1) &= s_1(1,\ q_j,\ 1,\ 1,\ \dots,\ 1) \\ &\quad + s_1(1,\ 1,\ r_j^{(1)},\ 1,\ \dots,\ 1) \\ &= B\log q_j + D_1 \log r_j^{(1)} \end{aligned}$$

$\therefore$ By Mathematical induction

$$s_1(1, q_j, r_j^{(1)}, \dots, r_j^{(n)}) = B\log q_j + D_1 \log r_j^{(1)} + \dots + D_n \log r_j^{(n)} \tag{2.2}$$

Next we apply (iii) with $l = n$, $p_{1j} = p_j$, $p_{2j} = 0 = p_{3j} = \ldots \ldots \ldots \ldots = p_{nj}$,

$$q_j = 1 = r_j^{(1)} = \ldots = r_j^{(n)}, \quad j = 1, 2, \ldots, m.$$

Then

$$s_{nm}\begin{bmatrix} p_1, & p_2, & \ldots, & p_m, & 0, & \ldots, & 0, & \ldots, & 0, & \ldots, & 0 \\ q_{11}, & q_{12}, & \ldots, & q_{1m}, & q_{21}, & \ldots, & q_{2m}, & \ldots, & q_{n1}, & \ldots, & q_{nm}, \\ r_{11}^{(1)}, & r_{12}^{(1)}, & \ldots, & r_{1m}^{(1)}, & r_{21}^{(1)}, & \ldots, & r_{2m}^{(1)}, & \ldots, & r_{n1}^{(1)}, & \ldots, & r_{nm}^{(1)}, \\ \\ r_{11}^{(n)}, & r_{12}^{(n)}, & \ldots, & r_{1m}^{(n)}, & r_{21}^{(n)}, & \ldots, & r_{2m}^{(n)}, & \ldots, & r_{n1}^{(n)}, & \ldots, & r_{nm}^{(n)}, \end{bmatrix}$$

$$= s_m\begin{bmatrix} p_1, & \ldots, & p_m \\ 1, & \ldots, & 1 \\ 1, & \ldots, & 1 \\ \\ 1, & \ldots, & 1 \end{bmatrix} + \sum_{j=1}^{m} p_j s_n \begin{bmatrix} 1, & 0, & \ldots, & 0 \\ q_{1j}, & q_{2n}, & \ldots, & 0_{nj} \\ r_{1j}^{(1)}, & r_{2j}^{(1)}, & \ldots, & r_{nj}^{(1)} \\ \\ r_{1j}^{(n)}, & r_{2j}^{(n)}, & \ldots, & r_{nj}^{(n)} \end{bmatrix}$$

By repeated application of (ii), this becomes

$$s_m\begin{bmatrix} p_1, & \ldots, & p_m \\ q_{11}, & \ldots, & q_{1m} \\ r_{11}^{(1)}, & \ldots, & r_{1m}^{(1)} \\ \\ r_{11}^{(n)}, & \ldots, & r_{1m}^{(n)} \end{bmatrix} = s_m\begin{bmatrix} p_1, & \ldots, & p_m \\ 1, & \ldots, & 1 \\ 1, & \ldots, & 1 \\ \\ 1, & \ldots, & 1 \end{bmatrix}$$

$$+ \sum_{j=1}^{m} p_j \, s_1(1\,;\, q_{1j}, r_{1j}^{(1)}, \ldots, r_{1j}^{(n)}) \tag{2.3}$$

Put $q_{1j} = q_j$, $r_{1j}^{(1)} = r_j^{(1)}$, $\ldots$, $r_{1j}^{(n)} = r_j^{(n)}$ in (2.3).

Then

$$s_m\begin{bmatrix} p_1, & \ldots; & p_m \\ q_1, & \ldots, & q_m \\ r_1^{(1)}, & \ldots, & r_m^{(1)} \\ \\ r_1^{(n)}, & \ldots, & r_m^{(n)} \end{bmatrix} = s_m\begin{bmatrix} p_1, & \ldots, & p_m \\ 1, & \ldots, & 1 \\ 1, & \ldots; & 1 \\ \\ 1, & \ldots, & 1 \end{bmatrix}$$

$$+ \sum_{j=1}^{m} p_j \cdot (D \log q_j + D_1 \log r_j^{(1)} + \cdots + D_n \log r_j^{(n)}) \tag{2.4}$$

Therefore it suffices to find out

$$d_m(p_1, \cdots, p_m) = s_m \begin{bmatrix} p_1, & \cdots, & p_m \\ 1, & \cdots, & 1 \\ 1, & \cdots, & 1 \\ & & \\ 1, & \cdots, & 1 \end{bmatrix}$$

From (i)–(iii) we obtain the following conditions on d_m:

(i)^1 $d_m(p_1, \ldots, p_m\}$ is a symmetric function of its variables

(ii)^1 $d_m(p_1, \ldots, p_{m-1}, 0) = d_{m-1}(p_1, \cdots, p_{m-1})$

(iii)^1 $d_{lm}(p_{11}, \cdots, p_{lm}) = d_m(p_1, \ldots, p_m)$

$$+ \sum_{j=1}^{m} p_j \, d_l(p_{1j}/p_j, \cdots, p_{lj}/p_j)$$

where p_j and p_{lj} are related as in (iii).

It has been shown by Faddeev (1956) that the only continuous functions which satisfy (i)^1—(iii)^1 are given by

$$d_m(p_1, \ldots, p_m) = A \sum_{j=1}^{m} p_j \log p_j,$$

where A is a real number and $0 \log 0 = 0$. Thus if we substitute this in (2.4) we get (2.1).

It can be easily verified that s_m as given by (2.1) satisfies postulates (i)—(iv) of the theorem.

3. Application to Information Theory: n th order I. I. is an information theoretic measure associated with $n+2$ probability distributions and the following theorem gives a characterisation of this measure.

THEOREM: *The information-theoretic measure (2.1) associated with $n+2$ probability distributions p_j, q_j, $r_j^{(1)}, \cdots, r_j^{(n)}$. $j = 1, 2, \ldots, m$ under the conditions*

(1) $s_2(p^1; p^1; p^1; \ldots; p^1) = 0$; $p^1 = (p, 1-p)$

(2) $s_2(1, 0; 1/2; 1/2; 1/2; 1/2; \ldots; 1/2; 1/2) = 0$

(3) $s_2(1, 0; 1, 0; 1, 0; \ldots; 1, 0; 1/2, 1/2; 1, 0) = (n^n - 1)$

(4) $s_2(1, 0; 1, 0; 1, 0; \ldots; 1, 0; 1/2, 1/2; 1, 0; 1, 0) = -(n^n - 2)$

(n) $s_2(1, 0; 1, 0; 1, 0; 1/2, 1/2; 1, 0; \ldots; 1, 0) = (-1)^{n-1}(2^n)$

($n+1$) $s_2(1, 0; 1, 0; 1/2, 1/2; \ldots; 1, 0) = (-1)^n (1^n)$

($n+2$) $s_2(1, 0; 1/2, 1/2; 1, 0; \ldots; 1, 0) = (-1)^{n+1} (0^n)$

is the n-th order I. I. given by (1.1).

Proof : (2.1 with the help of above $n+2$ conditions gives respectively

$$\begin{array}{lllll}
A+B+D_1+D_2+ & \cdots & \cdots & +D_n & = 0 \\
\quad B+D_1+D_2+ & \cdots & \cdots & +D_n & = 0 \\
 & & -D_{n-1} & & = (n^n-1) \\
 & -D_{n-2} & & & = -(n^n-2) \\
 & -D_2 & & & = (-1)^{n-1}\,(2^n) \\
-D_1 & & & & = (-1)^{n+1}\,(1^n) \\
-B & & & & = (-1)^{n+1}\,(0^n)
\end{array}$$

These give $A=0,\ B=(-1)^n\,(0^n),\ D_1=(-1)^{n-1}\,(1^n),$

$D_2=(-1)^{n-2}\,(2^n),\ \ldots\ D_{n-2}=(n^n-2),\ D_{n-1}=-(n^n-1),$

$D_n=(n^n)$

Substituting these values of constants in (2.1) we get (1.1)

Conclusions : In the present paper, possible applications of information improvement of higher orders have been hinted at, actual applications are contemplated to be discussed in future papers.

Acknowledgement : The author is grateful to Prof. M. Dutta, Professor-in-Charge, S. N. Bose Institute of Physical Science, Calcutta University for his valuable guidance and help.

The author is thankful to U. G. C. for the award of a Senior Research Fellowship.

Thanks are also due to the referee for his precious comments.

References

Aczel, J. (1966) : Lectures on Functional Equations and their Applications, Academic Press, New York.

Faddeev, D. K. (1956) : On the notion of entropy in a finite probability space, *Uspehi Mat. Nauk*, **11**, 227.

Roy, D. (1978) : One some information improvement, *Journal of Mathematical Science*, India (To Appear).

Theil, H. (1967) : Economics and Information Theory, North Holland Pub. Co., Amsterdam.

S. N. BOSE INSTITUTE OF PHYSICAL SCIENCES,
92, A. P. C. ROAD, CALCUTTA-700 009, INDIA.

Bull. Cal. Math. Soc., 75, 169—174 (1983)

ON STEINHAUS PROPERTY FOR DISTANCE SETS IN A METRIC SPACE

ARATI BISWAS and K. C. RAY

(*Received 18 April 1979*)

If (E, ρ) is a metric space with a measure μ defined on E, then the following property is referred to as the ***Steinhaus property for distance sets***:

If A is a measurable subset of E such that $\mu(A)>0$, then the distance set of A

$$D(A)=\{\rho(x, y) : x \in A \text{ and } y \in A\}$$

contains an interval with the origin as an end point.

E. Boardman (1972) has proved that under certain conditions on the metric space (E, ρ) all simple rectifiable curves in E have the Steinhaus property for distance sets.

A simple curve $C \subset E$ is the image of continuous injective mapping

$$f : [0, 1] \to E;$$

and if the measure of C is finite then C is rectifiable.

Let (E, ρ) be a metric space and let Λ denote linear measure which is the restriction of the metric outer measure Λ^* to the measurable sets (which include all Borel sets), where for $A \subset E$

$$\Lambda^*(A)=\sup_{\delta>0}\left\{\inf \left\{\sum_{i=1}^{\infty} d(A_i) : A_i \subset E, d(A_i)<\delta \text{ and } \bigcup_{i=1}^{\infty} A_i \supset A\right\}\right\},$$

diameter of A_i being denoted by $d(A_i)$.

Let $B \subset C$, and $r>0$. Then

$$B(r)=\{z \in C : \exists\, u \in B \text{ such that } u<z \text{ and } \rho(u, z)=r\},$$

$$B(-r)=\{z \in C : \exists\, u \in B \text{ such that } z<u \text{ and } \rho(u, z)=r\}.$$

Let (E, ρ) be a metric space, $C \subset E$ be a simple rectifiable curve determined by continuous injective map $f : [0, 1] \to E$, so that $C=f[0, 1]$.

The mapping f induces a linear ordering on the curve C such that for $x, y \in C$.

$$x<y \text{ if and only if } f^{-1}(x)<f^{-1}(y).$$

For $a, b \in C$ and $a<b$ $<a, b>$ denotes a subarc of C given by

$$<a, b>=\{ c \in C : a \leqslant c \leqslant b\}.$$

Then $<a, b>$ is a simple rectifiable curve.

Also for a subarc $<a,b>$ of C,

$$\wedge(<a, b>)=l\,(<a, b>) \tag{1}$$

where $l\,(<a, b>)$ denotes the length of the arc $<a, b>$, that is,

$$l\,(<a, b>)=\sup \left\{\sum_{r=1}^{n} \rho\,(x_{r-1}, x_r)\right\},$$

where the supremum is taken over all finite subdivisions $\{x_0, x_1, \cdots, x_n\}$ with

$$a=x_0<x_1<\ldots<x_n=b \text{ of } <a, b>.$$

Again if $b \in C$, $a_n \in C$, $a_n<a_{n+1}<b$ and $\lim \rho\,(a_n, b)=0$ then

$$\lim_{n\to\infty} l\,(<a_n, b>)=\lim_{n\to\infty} \wedge\,(<a_n, b>). \tag{2}$$

Then with the above considerations Boardman has proved (Proposition 1) the following result:

Let C be a simple rectifiable curve in a metric space (E, ρ). Suppose C has the following property:

there exist $c>0$ and $d_0>0$ such that for each subset $B \subset C$,

$$0<r<d_0 \text{ implies } d\,(B) \geqslant c\,\{d\,(B\,(-r))\}.$$

Then if S is a linearly measurable subset of C and $\wedge\,(S)>0$, $D\,(S)$ contains an interval with the origin as an end point. That is C has the Steinhaus property for distance sets.

In fact, in proving the above result Boardman has proved that $S \cap S(r) \neq \Phi$, where $0<r<\eta$ for some η.

In this paper (Theorem 1) following the same line of arguments as Boardman we have proved that the conditions of the above proposition admit of a more general result, namely, the measure of the set $S \cap S(r)$ is positive. Theorems 2 and 3 appear as applications of Theorem 1. In the last theorem (Theorem 4) we have extended the result of Theorem 1. In fact it has been shown that under certain conditions on E there exists an interval $(0, \eta)$ such that for any $r(0<r<\eta)$ the measure of the set $S \cap S(r)$ is arbitrarily equal to the measure of the given set S. This result may be compared to that of Theorem 1 of Mookhopadhyaya and Lahiri (1963) wherein the authors obtain a similar result for sets of positive measure in R_N (N-dimensional Euclidean space).

In this paper we have used the same notations as in (1972).

THEOREM 1. *Let C be a simple rectifiable curve in a metric space (E, ρ). Suppose C has the following property: there exist $c > 0$ and $d_0 > 0$ such that for each subset $B, \subset C, 0 < r < d_0$ implies $d(B) \geqslant c\{d(B(-r))\}$.*

Then if S is a measurable subset of C and $\wedge(S) > 0$ then there exists an interval $(0, \eta)$ such that for every r belonging to this interval $S \cap S(r)$ is of positive measure.

Proof: In view of (2) we assume that dist $(f(1), S) > 0$. Since S is measurable and C is rectifiable there exists a compact set K and an open set G, both being subsets of C such that

$$K \subset S \subset G \subset C,$$

$$f(1) \in C \setminus G$$

and

$$\wedge(K) > (1 \setminus c) \wedge (G \setminus K),$$

which may be deduced from Theorem 13.5 of (Munroe 1959) as indicated by Boardman.

Then proceeding in the same manner as in Proposition 1 of (1972) we see that there axists $\eta\,(> 0)$ such that for any $r\,(0 < r < \eta)$,

$$K(r) \subset G.$$

Also as deduced by Boardman we have

$$\wedge(K(r)) > \wedge(G \setminus K).$$

If $X = K \cap K(r)$, then

$$X = G \setminus [(G \setminus K) \cup (G \setminus K(r))].$$

So,

$$\begin{aligned} \wedge(X) &\geq \wedge(G) - [\wedge(G \setminus K) + \wedge(G \setminus K(r))] \\ &> \wedge(G) - [\wedge(K(r)) + \wedge(G) - \wedge(K(r))] \\ &= 0. \end{aligned}$$

Thus $\wedge(K \cap K(r)) > 0$. Since $K \subset S$, $K(r) \subset S(r)$ for $r \in (0, \eta)$, and hence

$$\wedge(S \cap S(r)) > 0 \text{ for } r \in (0, \eta).$$

THEOREM 2. *Let C be a simple rectifiable curve in a metric space (E, ρ) satisfying the conditions of Theorem 1. If S is a measurable subset of C and $\wedge(S) > 0$ then for any positive integer p there exist distinct positive numbers $r_1, r_2, \cdots, r_p$ such that*

$$S \cap S(r_1) \cap S(r_2) \cap \cdots \cap S(r_p)$$

is a set of positive measure.

Proof. Since S is measurable and C is rectifiable by Theorem 1 there exists a positive number r_1 such that

$$\wedge[S \cap S(r_1)] > 0.$$

Let $S \cap S(r_1) = S_1$. Then since S_1 is a set of positive measure and $S_1 \subset S$, by Theorem 1 there exists a positive number $r_2 \neq r_1$ such that

$$\wedge[S_1 \cap S_1(r_2)] > 0,$$

where
$$S_1 \cap S_1(r_2) = S \cap S(r_1) \cap S_1(r_2)$$
$$\subset S \cap S(r_1) \cap S(r_2).$$

Proceeding in this way after a finite number of steps we obtain a positive number r_p distinct from $r_1, r_2, \dots r_{p-1}$ and a set $S_{p-1}' \subset S$ such that
$$\Lambda\,(S_{p-1} \cap S_{p-1}(r_p)) > 0.$$

Now,
$$S_{p-1} \cap S_{p-1}(r_p) = S \cap S(r_1) \cap S_1(r_2) \cap \cdots \cap S_{p-1}(r_p)$$
$$\subset S \cap S(r_1) \cap S(r_2) \cap \cdots \cap S(r_p).$$

Hence the set $S \cap S(r_1) \cap S(r_2) \cap \dots \cap S(r_p)$ is of positive measure, where $r_1, r_2, \dots, r_p$ are all distinct positive numbers.

THEOREM 3. *Let C be a simple rectifiable curve in a metric space* (E, ρ) *satisfying the conditions of Theorem 1. Then if S is a measurable subset of C and* $\Lambda\,(S) > 0$, *then there exist distinct positive numbers* r_1, r_2 .. *and a point* $\alpha \in S$ *such that*
$$\alpha \in S \cap S(r_1) \cap S(r_2) \cap \cdots .$$

Proof. In proving Theorem 1 we have shown that there exist a compact set $K_1 \subset S$ and a positive number r_1 such that
$$\Lambda\,[\,K_1 \cap K_1(r_1)\,] > 0.$$

We denote the set $K_1 \cap K_1(r_1)$ by S_1. Since S_1 is a set of positive measure there exist a compact set $K_2 \subset S_1$ and a positive number $r_2 \neq r_1$ such that
$$\Lambda\,[\,K_2 \cap K_2(r_2)\,] > 0$$
where $K_2 \subset S_1 \subset K_1(r_1) \subset S(r_1)$.

We denote $K_2 \cap K_2(r_2)$ by S_2. Again, since S_2 is a set of positive measure there exist a compact set $K_3 \subset S_2$ and a positive number $r_3 \neq r_2 \neq r_1$ such that
$$\Lambda\,[\,K_3 \cap K_3(r_3)\,] > 0$$
where $K_3 \subset S_2 \subset K_2(r_2) \subset S(r_2)$.

Continuing this process indefinitely, we obtain a decreasing sequence of non-empty compact sets
$$K_1 \subset K_2 \subset K_3 \subset \dots \dots$$
Therefore there exists a point $\alpha \in S$ such that
$$\alpha \in K_1 \cap K_2 \cap K_3 \cap \dots \dots$$
But since $K_1 \subset S$, $K_2 \subset S(r_1)$, $K_3 \subset S(r_2)$, and so on,
$$\alpha \in S \cap S(r_1) \cap S(r_2) \cap \dots \dots$$
where $r_1 \neq r_2 \neq r_3 \neq \dots \dots$

THEOREM 4. *Let C be a simple rectifiable curve in a metric space* (E, ρ) *with the following property: there exist* $d_0 > 0$ *and* $\lambda > 0$ *such that for any sequence of* *subset* $B_1, B_2, \ldots$ *of* C *whose diameters are less then* λ,

$$0 < r < d \quad \text{implies} \sum_{i=1}^{\infty} d(B_i) \geqslant \sum_{i=1}^{\infty} d(B_i(-r)).$$

Then if S *is a linearly measurable subset of* C *of positive measure and* l *is any number* $(0 < l < \Lambda(S))$, *there exists an interval* $(0, \eta)$ *such that for any* $r(0 < r < \eta)$ *we have*

$$\Lambda(S \cap S(r)) > \Lambda(S) - l.$$

Proof. Let S be a measurable subset of C and $\Lambda(S) > 0$. Suppose C is determined by

$$f : [0, 1] \to C.$$

Then is view of (2) we can assume that dist $(f(1), S) > 0$.

Since S is measurable and C is rectifiable there exist a compact set K and an open (in C) set G such that for any number $l(0 < l < \Lambda(S))$ there exists a positive number $s > 2\Lambda(K)/l$ for which

$$K \subset S \subset G \subset C,$$

$$f(1) \in C \setminus G$$

and

$$\Lambda(K) > \frac{2s\,\Lambda(K)}{sl - 2\,\Lambda(K)}\,\Lambda(G \setminus K)$$

$$\text{i.e.,} \quad \Lambda(G \setminus K) < \frac{sl - 2\,\Lambda(K)}{2s}.$$

Let δ be a positive number less than $2\Lambda(K)/s$. Then there exists a positive number ϵ_0 such that for any ϵ with $0 < \epsilon < \epsilon_0$ all covers $\{A_i\}_{i=1}^{\infty}$ of K with $d(A_i) < \epsilon$ satisfy

$$\sum_{i=1}^{\infty} d(A_i) > \Lambda(K) - \delta. \tag{3}$$

Let $d_1 = \text{dist}\,(K, C \setminus G)$. Then $d_1 > 0$.

Let $\eta = \min\{d_0, d_1, 1/3\,\epsilon_0\}$. Then for any r with $0 < r < \eta$,

$$K(r) \subset G.$$

Let ϵ be any number satisfying $0 < \epsilon < 1/3\,\epsilon_0$ and $B_i \subset C$ be such that $d(B_i) < \min(\epsilon, \lambda)$ and

$$K(r) \subset \bigcup_{i=1}^{\infty} B_i. \tag{4}$$

Then as deduced by Boardman

$$\bigcup_{i=1}^{\infty} B_i(-r) \supset K$$

where $d(B_i(-r)) < \epsilon_0$.

It, therefore, follows from (3) that

$$\sum_{i=1}^{\infty} d(B_i(-r)) \geq \Lambda(K)-\delta.$$

Also since $r < \eta < d_0$ and $d(B_i) < \lambda$,

$$\sum_{i=1}^{\infty} d(B_i) \geq \sum_{i=1}^{\infty} d(B_i(-r))$$

so that

$$\sum_{i=1}^{\infty} d(B_i) \geqslant \Lambda(K)-\delta.$$

Hence it follows from (4) that

$$\Lambda(K(r)) \geqslant \Lambda(K)-\delta.$$

Let $X = K \cap K(r)$. Then

$$\begin{aligned}
\Lambda(X) &\geq \Lambda(G)-[\Lambda(G\setminus K)+\Lambda(G\setminus K(r))] \\
&= \Lambda(G)-[\Lambda(G)-\Lambda(K)+\Lambda(G)-\Lambda(K(r))] \\
&\geq \Lambda(G)-[2\Lambda(G\setminus K)+\Lambda(K)-\Lambda(K)+\delta] \\
&= \Lambda(G)-[2\Lambda(G\setminus K)+\delta] \\
&> \Lambda(G)-\left[2\frac{sl-2\Lambda(K)}{2s}+\frac{2\Lambda(K)}{s}\right] \\
&= \Lambda(G)-l \\
&\geqslant \Lambda(S)-l.
\end{aligned}$$

Since $S \cap S(r) \supset K \cap K(r)$, we have

$$\Lambda[S \cap S(r)] > \Lambda(S)-l.$$

This proves the theorem.

References

Boardman, E., (1972): On extensions of the Steinhaus theorem for distance sets and difference sets, *J. London Math. Soc.* (2), 5, 729.

Munroe, M. E., (1959): Introduction to measure theory and integration (Addison-Wesley).

Mookhopadhyaya, A. K. and Lahiri B. K. (1963): Some results on closed sets of positive measure, *Bull. Cal. Math. Soc.*, 55, 1, 37.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KALYANI
WEST BENGAL.

Bull. Cal. Math. Soc., 75, 175—178 (1983)

ON A THEOREM CONCERNING LEBESGUE-INTEGRATION

S. Ghosh

(Received 5 March 1979, 1st Revision Received 20 Dec. 1979, 2nd Revision Received 18 March, 1980)

Abstract : In this note, a theorem concerning change of variable in Lebesgue-integration is discussed.

1. Introduction : A theorem concerning change of variable in Lebesgue-integration has been proved in (F. Riesz and B. sz-Nagy (1954) P 55). Here the same theorem will be proved under a slightly modified set of conditions.

2. The Propositions : We state below the theorem, which will be proved here [for definitions etc. see (Rudin (1960))] directly, and improve the same as theorem 2 at the end.

THEOREM 1. *Let* $-\infty < a < b < \infty$. *Let* $g : [a, b] \to (-\infty, \infty)$ *be a non-decreasing continuous function with* $g(a) < g(b)$.

Let $f : [g(a), g(b)] \to (-\infty, \infty)$ be a Lebesgue-measurable right-continuous function $\ni f$ is integrable over $[g(a), g(b)]$ (with respect to Lebesgue-measure). We assume that the first order right-derivative g' of g exists as a real-valued right-continuous function on $[a, b]$.

Then

$$\int_a^b f(g(x))\, g'(x)\, dx$$

exists finitely and

$$\int_{g(a)}^{g(b)} f(x)\, dx = \int_a^b f(g(x))\, g'(x)\, dx$$

Proof 1. Let n be an integer ≥ 1. Write $a_0 = a$. Now, by the hypothesis of the theorem, $\exists\, b_0 \epsilon\, (a, b) \ni$ the following statement is true :

$$S_0 \left| \begin{array}{l} | f(g(x))\, g'(x) - f(g(a_0))\, g'(a_0) | \leqslant (2n)^{-1} \ \forall\, x \epsilon\, [a_0, b_0], \\ | f(g(a_0))\, g'(a_0) - f(g(a_0))\, (b_0 - a_0)^{-1} (g(b_0) - g(a_0)) | \leq (2n)^{-1}, \\ | f(x) - f(g(a_0)) | \leq (n)^{-1} \ \forall\, x \epsilon\, [g(a_0), g(b_0)] \end{array} \right.$$

Define $I_0 = [a_0, b_0)$.

Now again taking $a_1 = b_0$, $\exists\, b_1 \in (a_1, b) \ni$ the statement S_1 holds, where S_1 is the statement got by replacing 0 by 1 throughout in S_0.

Define $I_1 = [a_1, b_1)$.

(2) Suppose corresponding to every ordinal $\alpha < k < \omega$ [ω=first uncountable ordinal number], we have defined a subset I_α of $[a, b) \ni$ [for the literature of ordinal numbers, see Halmos (1960)].

(I) If for some ordinal $\alpha < k$, I_α be nonempty, then I_α is a left-closed right open non-degenerate interval $[a_\alpha, b_\alpha)$ where $a \leqslant a_\alpha < b_\alpha < b$ and the statement S_α holds where S_α is the statement got by replacing 0 by α throughout in S_0.

(II) if $\alpha < \beta < k$ and I_β be nonempty, then

(a) I_α is nonempty and

(b) $x_1 \in I_\alpha, x_2 \in I_\beta$ always implies $x_1 < x_2$, $\bigcup\limits_{\substack{p\,:\,p \text{ ordinal}\\ p \leqslant \beta}} I_p = [a_0, b_\beta)$;

(III) if $\alpha < \alpha + 1 < k$ and $I_{\alpha+1}$ is nonempty, then $b_\alpha = a_{\alpha+1}$.

If for some ordinal $\alpha < k$, I_α=empty set, then define I_k=empty set.

Suppose $\forall$ ordinal $\alpha < k$, I_α is nonempty.

Let t=supremum $\{b_\alpha, \alpha < k\}$. Then, $\bigcup\limits_{\substack{\alpha\,:\,\alpha \text{ ordinal}\\ \alpha < k}} I_\alpha = [a_0, t)$.

If $t = b$, then define I_k=empty set. Suppose $t < b$. Then taking $a_k = t$, $\exists\, b_k \in (a_k, b) \ni$ the statement S_k holds, where S_k is the statement got by replacing 0 by k throughout in S_0.

Define $I_k = [a_k, b_k)$. We have thus defined I_α for each ordinal $\alpha < \omega$.

(III) Since there cannot exist uncountably many pairwise disjoint Lebesgue-measurable subsets of $[a, b)$ with strictly positive Lebesgue-measure each, $\exists$ a first ordinal $k < \omega \ni I_k$=empty set.

Let $\{A_1, A_2, \ldots\}$ be an enumeration of $\{I_\alpha : \alpha < k\}$ as pairwise disjoint sets, where $A_i = [c_i, d_i)$, $i \geqslant 1$. Let $B_i = [g(c_i), g(d_i)]$, $i \geqslant 1$.

For each integer $i \geqslant 1$ and $\forall\, x \in A_i$,

$$\Big|\, |f(g(x))\, g'(x)| - |f(g(c_i))\,(d_i - c_i)^{-1}\,(g(d_i) - g(c_i))| \,\Big|$$

$$\leqslant |f(g(x))\, g'(x) - f(g(c_i)\,(d_i - c_i)^{-1}\,(g(d_i) - g(c_i))| \leqslant n^{-1}.$$

Hence for each integer $i \geqslant 1$,

$$\left| \int_{A_i} f(g(x))\, g'(x)\, dx - f(g(c_i))\,(g(d_i) - g(c_i)) \right| \leqslant n^{-1}\,(d_i - c_i) \tag{1}$$

$$\left| \int_{A_i} |f(g(x))|\, g'(x)\, dx - |f(g(c_i))|\,(g(d_i) - g(c_i)) \right| \leqslant n^{-1}\,(d_i - c_i) \tag{2}$$

Since $\forall$ integer $i \geqslant 1$ and $\forall\, x \in [g(c_i), g(d_i)]$,

$$\| f(x)| - |f(g(c_i))\| \leqslant |f(x)-f(g(c_i))| \leqslant n^{-1},$$

We have that

$$\left|\int_{B_i} |f(x)|\,dx - |f(g(c_i))|\,(g(d_i)-g(c_i))\right| \leqslant n^{-1}\,(g(d_i)-g(c_i)) \tag{3}$$

$$\left|\int_{B_i} f(x)\,dx - f(g(c_i))\,(g(d_i)-g(c_i))\right| \leqslant n^{-1}\,(g(d_i)-g(c_i)) \tag{4}$$

Summing (3) over all $i=1, 2, \ldots$ we get that,

$$\sum_{i=1}^{\infty} |f(g(c_i))|\,(g(d_i)-g(c\ )) \leqslant \sum_{i=1}^{\infty}\left[\int_{B_i} |f(x)|\,dx + n^{-1}\,(g(d_i)-g(c_i))\right]$$

$$= \int_{g(a)}^{g(b)} |f(x)|\,dx + n^{-1}\,(g(b)-g(a)) < \infty.$$

Hence

$$\sum_{i=1}^{\infty} f(g(c_i))\,(g(d_i)-g(c_i))$$

converges absolutely and, from (4)

$$\left|\int_{g(a)}^{g(b)} f(x)\,dx - \sum_{i=1}^{\infty} f(g(c_i))\,(g(d_i)-g(c_i))\right| \leqslant n^{-1} \sum_{i=1}^{\infty}\,[g(d_i)-g(c_i)]$$

$$= n^{-1}\,(g(b)-g(a))$$

Using monotone convergence theorem and (2)

$$\int_a^b |f(g(x))|\,g'(x)\,dx = \sum_{i=1}^{\infty} \int_{A_i} |f(g(x))|\,g'(x)\,dx$$

$$\leqslant \sum_{i=1}^{\infty}\,[\,|f(g(c_i))|\,(g(d_i)-g(c_i)) + n^{-1}\,(d_i-c_i)]$$

$$\leqslant \sum_{i=1}^{\infty} |f(g(c_i))|\,(g(d_i)-g(c_i)) + n^{-1}\,(b-a) < \infty.$$

Hence

$$\int_a^b f(g(x))\,g'(x)\,dx$$

exists finitely and, from (1),

$$\left|\int_a^b f(g(x))\,g'(x)\,dx - \sum_{i=1}^{\infty} f(g(c_i))\,(g(d_i)-g(c_i))\right|$$

$$\leqslant n^{-1} \sum^{\infty} (d_i-c_i) = n^{-1}\,(b-a) \tag{6}$$

Combining (5) and (6) we get that.

$$\left| \int_a^b f(g(x))\, g'(x)\, dx - \int_{g(a)}^{g(b)} f(x\, dx \right|$$
$$\leqslant n^{-1}\,[b-a+g(b)-g(a)] \to 0 \text{ as } n \to \infty. \text{ Q.E.D.}$$

Combining theorem 1 with a theorem in [Rudin (1960) P 165], we get the following corollary to theorem 1.

Corollary : The g of theorem 1 is absolutely continuous on $[a, b]$

Now, by virtue of known result cited in §1, we have

THEOREM 2 : *Theorem 1 holds without the right continuity requirement on f.*

Acknowledgement : Thanks are due to the referee for constructive suggestions.

References

Halmos, P. R. (1960) : "Naive Set-theory" Van-Nostrand.

Riesz, F. and Nagy, B. (1954) : "Functional analysis" Blackie and Sons Ltd.

Rudin, Walter (1960) : "Real and Complex analysis" Mcgraw Hill.

SATYENDRANATH BOSE INSTITUTE OF
PHYSICAL SCIENCES AND
DEPARTMENT OF STATISTICS,
PRESIDENCY COLLEGE,
CALCUTTA-700073.

Bull. Cal. Math. Soc., 75, 179—184 (1983)

SPACES OF STRONGLY σ-SUMMABLE SEQUENCES

S. K. Saraswat and S. K. Gupta

(*Received 28 April 1979*)

Abstract : Maddox [1967, 1970] introduced the concept of strongly summable sequences and recently Nanda [1976] has defined the spaces of strongly almost summable sequences using the concept of almost convergence [1948]. In this paper we investigate some new sequence spaces by using the definition of invariant mean and prove such spaces to be complete paranormed spaces under certain conditions, some topological results, and characterization of strongly σ-regular matrices have also been discussed.

1. **Introduction** : Let l_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x = \{ x_k \}$ with the usual norm $\| x \| = \sup_k | x_k |$.

Let σ be a mapping of the set of positive integers into itself. A continuous linear function Φ on l_∞ is said to be an invariant mean or σ-mean if and only if (i) $\Phi(x) \geqslant 0$ whenever $x_n \geqslant 0$ for all n, (ii) $\Phi(e) = 1$, where $e = \{ 1, 1, 1 \,.. \}$, and (iii) $\Phi(\{ x_{\sigma(n)} \}) = \Phi(x)\ \forall\ x \in l_\infty$. Let V_σ denote the space of bounded sequence all of whose σ-mean are equal.

If $x = \{ x_n \}$. write $Tx = \{ x_{\sigma(n)} \}$. It is easy to show that the V_σ can be characterized as the set of all bounded sequences x for which $\lim\limits_m (x+Tx+ \ldots +T^m x)/(m+1)$ exists in l_∞ and has the form Le, $L = \sigma - \lim x$. Recently Schaefer [1972] has characterized some matrics in V_σ.

2. **Preliminaries** : If p_k is real such that $p_k > 0$ and $\sup_k p_k < \infty$ (Maddox 1967). Let $a(n, k)$ denote the element a_{nk} of the matrix A, for $m \geqslant 0$, we have

$$(Ax+TAx+ \ldots +T^m Ax) \,/\, (m+1)$$

$$\left\{ \sum_k \left[a\,(n, k) + a\,(\sigma_{(n)}, k) + \ldots + a\left(\sigma^m_{(n)}, k\right) \right] x_k/(m+1) \right\}_{n=1}^{\infty}$$

where $\sigma^m(n)$ denotes the mth iterate of σ at n. For every $n \geqslant 1$, we write

$$T_{m,n}(x) = \sum_k \sum_{j=0}^{m} a(\sigma^j_{(n)}, k) \mid x_k \mid^{p_k} / \,(m+1),$$

$$= \sum_k \alpha(n, k, m) \mid x_k \mid^{p_k}.$$

where

$$\alpha(n, k, m) = \sum^{m} a(\sigma^j_{(n)}, k) \,/\, (m+1).$$

We define

$$[\,A_\sigma, p\,]_0 = \{\, x : T_{m,n}(x) \to 0 \text{ uniformly in } n \,\},$$

$$[\,A_\sigma, p\,] \;= \{\, x : T_{m,n}(x - Le) \to 0 \text{ for some } L \text{ uniformly in } n \,\},$$

and

$$[\,A_\sigma, p\,]_\infty = \{\, x : \sup_{m,n} T_{m,n}(x) < \infty \,\},$$

which we call the space of strongly σ-summable to zero, strongly σ-summable and strongly σ-bounded sequences respectively.

If $\sigma(n) = n+1$, we have (Nanda 1976),

$$[\,A_\sigma, p\,]_0 = [\,\hat{A}, p\,]_0,\ [\,A_\sigma, p\,] = [\,\hat{A}, p\,], \text{ and } [A_\sigma, p]_\infty = [\,\hat{A}, p\,]_\infty .$$

Definition. If $x \in [A_\sigma, p]$ with limit L, we write $x_k \to L\,[A_\sigma, p]$. A pair (A, p) will be called strongly σ-regular if

$$x_k \to L \Rightarrow L\,[\,A_\sigma, p\,].$$

3. Main Results : THEOREM 1 : *Let* $p \in l_\infty$. *Then* $[\,A_\sigma, p\,]_0$ *and* $[\,A_\sigma, p\,]_\infty$ $(\inf p_k > 0)$ *are complete linear topological spaces paranormed by* $h(x) = \sup_{m,n} |\, T_{m,n}(x)\,|^{1/M}$, *where* $M = \max(1, \sup p_k)$ *if,*

$$\|\, A \,\| = \sup_m \sum_k \alpha(n,\, k,\, m) < \infty, \tag{3.1}$$

$$\sum_k \alpha(n,\, k,\, m) \to 0 \text{ uniformly in } n, \tag{3.2}$$

then $[\,A_\sigma, p\,]$ *has the same property. If further* $p_k = p\ \forall\ k$, *they are Banach spaces for* $1 \le p < \infty$ *and p-normed spaces for* $0 < p < 1$.

We shall require the following lemmas in the sequel.

4. Lemmas : LEMMA 1 : *If* $p \in l_\infty$, *then* $[\,A_\sigma, p\,]_0$, $[\,A_\sigma, p\,]$ *and* $[\,A_\sigma, p\,]_\infty$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof : First we consider $[\,A_\sigma, p\,]$. If $H = \sup p_k$ and $K = \max(1, 2^{H-1})$, we have (Maddox 1967 p. 346)

$$|\, x_k + y_k \,|^{p_k} \le K\,(\,|\, x_k \,|^{p_k} + |\, y_k \,|^{p_k}) \tag{a}$$

and for $\lambda \in \mathbb{C}$.

$$|\, \lambda \,|^{p_k} \leqslant \max(1,\ |\, \lambda \,|^{H}). \tag{b}$$

Suppose that $x_k \to L\,[\,A_\sigma, p\,]$, $y_k \to L'\,[\,A_\sigma, p\,]$ and $\lambda, \mu \in \mathbb{C}$. Then we have

$$T_{m,n}\,(\lambda x + \mu y - (\lambda L + \mu L')\,e\,) \leqslant KK'\, T_{m,n}\,(x - Le) + KK''\, T_{m,n}\,(y - L'e),$$

where $K' = \sup |\, \lambda \,|^{p_k}$ and $K'' = \sup |\, \mu \,|^{p_k}$ and this implies that

$$\lambda x + \mu y \to (\lambda L + \mu L')\,[\,A_\sigma, p\,].$$

This terminates the proof.

Now $[A_\sigma, p]$ and $[A_\sigma, p]_\infty$ also can be treated similarly.

LEMMA 2. $[A_\sigma, p] \subset [A_\sigma, p]_\infty$ *if* (3.1) *holds.*

Proof: Suppose that $x_k \to L\,[A_\sigma, p]$ and (3.1) holds. Now by using inequality (a).

$$\begin{aligned} T_{m,n}(x) &= T_{m,n}(x-Le+Le) \\ &\leqslant K\,T_{m,n}(x-Le)+K\sum_k \alpha(n,k,m)\,|\,L\,|^{p_k}, \qquad (c) \\ &\leqslant K\,T_{m,n}(x-Le)+K(\sup|\,L\,|^{p_k})\sum_k \alpha(n,k,m). \end{aligned}$$

Therefore $x \in [A_\sigma, p]_\infty$ and this completes the proof.

LEMMA 3: $[A_\sigma, p]_0$ *and* $[A_\sigma, p]_\infty$ *are complete with respect to their paranorm topologies.*

Proof: Let $\{x^i\}$ be a Cauchy sequence in $[A_\sigma, p]_0$. Then $\exists$ a sequence x such that $h(x^i-x) \to 0 \ (i \to \infty)$. Since h is subadditive it follows that $x \in [A_\sigma, p]_0$. The completeness of $[A_\sigma, p]_\infty$ can be similarly obtained.

5. Proof of Theorem 1: One can easily see that $h(0)=0$ and $h(-x)=h(x)$. Since $M \geqslant 1$, by Minkowski's inequality it follows that h is subaddition. It remains to show that the scalar multiplication is continuous. It follows from (b) that for $\lambda \in \mathbf{C}$ and $x \in [A_\sigma, p]_0$,

$$h(\lambda x) \leqslant \sup|\,\lambda\,|^{p_k/M}\,h(x).$$

Therefore, $x \to 0 \Rightarrow \lambda x \to 0$ (for fixed λ).

Now let $\lambda \to 0$ and x be fixed, given $\epsilon > 0$, $\exists N$ such that

$$T_{m,n}(\lambda x) < \epsilon/2. \quad (\forall\, n,\ \forall\, m > N), \qquad (d_1)$$

since $T_{m,n}(x)$ exists for all m, we write

$$T_{m,n}(x) = K(m) \qquad (1 \leq n \leqslant N),$$

and

$$\delta = \left(\frac{\epsilon}{2K(m)}\right)^{1/p_k}.$$

Then for $|\,\lambda\,| < \delta$,

$$T_{m,n}(\lambda x) > \epsilon/2\ (\forall\, n,\ 1 \leq m \leq N). \qquad (d_2)$$

It follows from (d_1) and (d_2) that

$$\lambda \to 0 \Rightarrow \lambda x \to 0 \quad (x \text{ fixed}).$$

This shows that $[A_\sigma, p]_\sigma$ is linear topological space paranormed by h.

Now if $\inf p_k = \theta > 0$ and $0 < |\lambda| < 1$ then $\forall\, x \in [A_\sigma, p]_\infty$

$$h^M(\lambda x) \leqslant |\,\lambda\,|\,\theta\,h^M(x).$$

Therefore $(A_\sigma, p]_\infty$ has the paranorm h, if (3.1) holds, it is clear from Lemma 2 that $h(x)$ exists for each $x \in [A_\sigma, p]$.

If (3.1) holds and $\{x^i\}$ is a Cauchy sequence in $[A_\sigma, p]$. Then $\exists x$ such that $h(x^i - x) \to 0$. If (3.2) holds, then from inequality (c) it is clear that $[\cdot A_\sigma, p] = [A_\sigma, p]_0$. Hence by Lemma 3, it follows immediately that $[A_\sigma, p]$ is complete linear topological space paranormed by h.

This completes the proof.

Remarks: It is evident that h is not a norm in general. But if $p_k = p \, \forall k$, then clearly h is a norm for $1 \leq p < \infty$ and a p-norm for $0 < p < 1$. Hence $[A_\sigma, p]_0$, $[A_\sigma, p]$ and $[A_\sigma, p]_\infty$ are Banach spaces for $1 \leq p < \infty$ and p-normed spaces for $0 < p < 1$.

Theorem 2: *Let $0 < p_k \leq 1$. Then $[A_\sigma, p]_0$ and $[A_\sigma, p]_\infty$ are locally bounded in $\inf p_k > 0$. If (3.1) holds then $[A_\sigma, p]$ has the same property.*

Proof: We shall only prove for $[A_\sigma, p]_\infty$. Let $\inf p_k = \theta > 0$. If $x \in [A_\sigma, p]_\infty$, then $\exists$ a constant $K' > 0$ such that

$$\sum_k \alpha (n, k, m) \mid x_k \mid^{p_k} \leq K' \; (\forall \; m, n)$$

For this K' and given $\delta > 0$, choose an integer $N > 1$ such that

$$N^\theta \geqslant K' / \delta$$

Since $N > 1$ and $p_k \geqslant \theta$ we have

$$\frac{1}{N^{p_k}} \leqslant \frac{1}{N^\theta} \quad (\forall_k).$$

Then for all m and n, we get

$$\sum_k \alpha (n, k, m) \left| \frac{x_k}{N} \right|^{p_k} \leqslant \frac{1}{N^\theta} \sum_k \alpha (n, k, m) \mid x_k \mid^{p_k}$$

$$\leqslant \frac{K'}{N^\theta} \leqslant \delta.$$

Therefore by taking supremum over m and n, we get

$$\{x : h (x) \leqslant K'\} \subseteq N \{x : h (x) \leqslant \delta\}.$$

For every $\delta > 0$, there exists $N > 1$ for which the above inclusion holds and so

$$\{x : h (x) \leqslant K'\}$$

is bounded.

This terminates the proof.

Remark: It is known that every locally bounded linear topological space is r-convex for some r such that $0 < r \leqslant 1$. Hence we have $[A_\sigma, p]_0$, and $[A_\sigma, p]_\infty$ are r-convex for all r, where $0 < r < \lim \inf p_k$.

We now characterize the class of strongly σ-regular matrices.

THEOREM 3 : *Let $0 < \theta \leqslant p_k \leqslant H < \infty$. The (A, p) is strongly σ-regular if and only if $Ax \in V_\sigma$ with σ-lim $x=0$ for all $x \in c_0$.*

Proof : *Necessity.* Suppose that (A, p) is strongly σ-regular. Therefore

$$| x_k - L |^{1/p_k} \to 0 \Rightarrow \sum_k \alpha (n, k, m) | x_k - L | \to 0$$

uniformly in n. Again since

$$\frac{1}{p_k} \geqslant \frac{1}{H} > 0,$$

therefore we have

$$x_k \to L \Rightarrow | x_k - L |^{1/p_k} \to 0.$$

[Also we have, If $p_k, q_k > 0$, then $c_0(q) \subset c_0(p) \Longleftrightarrow \liminf \dfrac{p_k}{q_k} > 0$ (Maddox 1967, p. 347)].

Thus

$$x_k \to L \Rightarrow \sum_k \alpha (n, k. m) (x_k - L) \to 0$$

uniformly in n and hence $Ax \in V_\sigma$ with σ-lim $x=0$ for all $x \in c_0$.

Sufficiency. Since $p_k \geqslant \theta > 0$ we have

$$x_k \to L \Rightarrow | x_k - L |^{p_k} \to 0.$$

Again we have $Ax \in V_\sigma$ with σ-lim $x=0$ for all $x \in c_0$. Therefore $x_k \to L\,[A_\sigma, p]$.

This completes the proof.

THEOREM 4 : *Suppose that $A x \in V_\sigma$ with σ-lim $x=0$ for all $x \in c_0$ and $p=\{p_k\}$ converges to a positive limit. Then $x=\{x_k\} \to L \Rightarrow x_k \to L\,[A_\sigma, p]$ uniquely if and only if*

$$\sum_k \alpha (n, k, m) \nrightarrow 0 \text{ uniformly in } n. \tag{4.1}$$

Proof : *Necessity.* Suppose that $Ax \in V_\sigma$ with σ-limt zero for all null sequences and $\{p_k\}$ be bounded. Let $x_k \to L \Rightarrow x_k \to L\,[A_\sigma, p]$ uniquely. We have $e \to 1\,[A_\sigma, p]$. Therefore the condition (4.1) must hold. For, otherwise $e \to 0\,[A_\sigma, p]$ which contradicts the uniqueness of L.

Sufficiency. Suppose that the condition (4.1) holds and $A x \in V_\sigma$ with σ-lim $x=0$ for all $x \in c_0$ and that $p_k \to g > 0$. Again suppose that $x_k \to L \Rightarrow x_k \to L\,[A_\sigma, p]$ and $x_k \to L'\,[A_\sigma, p]$ where $| L - L' | = t > 0$.

Then we have

$$\lim_{n\to\infty} \sum_k \alpha (n, k, m)\, U_k = 0 \text{ (uniformly in } n) \tag{4.2}$$

where

$$U_k = | x_k - L |^{p_k} + | x_k - L' |^{p_k}.$$

By the assumption we have $U_k \to t'$. Since $Ax \in V_\sigma$ with σ-lim $x=0$ for all $x \in c_0$, $U_k \to t'$ implies that

$$\sum_k \alpha(n, k, m) \mid U_k - t' \mid \to 0 \text{ (uniformly in } n) \tag{4.3}$$

But we have

$$t' \sum_k \alpha(n, k, m) \leqslant \sum_k \alpha(n, k, m)\, U_k + \sum_k \alpha(n, k, m) \mid U_k - t' \mid \tag{4.4}$$

Now by (4.2), (4.3) and (4.4) it follows that

$$\lim_{m\to\infty} \sum_k \alpha(n, k, m) = 0 \text{ (uniformly in } n).$$

Since this contradicts (4.1), we must have $L=L'$

This completes the proof.

Finally, we would like to express our sincere thanks to Dr. Z. U. Ahmed for his valuable suggestions.

References

Lorentz, G. G. (1948) : A contribution to the the theory of divergent sequences, *Acta Math.*, 80. 167.

Maddox, I. J. (1967) : Spaces of strongly summable sequences, *Quarterly J. Math. Oxford Ser.* (2) 18, 345.

— — (1970) : Elements of Functional Analysis, Camb. University Press.

Nanda, S. (1976) : Some sequence spaces and Almost Convergence, *J. Aust. Math. Soc.* 22 (A) 446.

Schaefer, P, (1972) : Infinite matrices and Invariant means, *Proc. Amer. Math. Soc.* 36 (1), 104.

DEPARTMENT OF MATHEMATICS,
ALIGARH MUSLIM UNIVERSITY,
ALIGARH-202001 (INDIA).

Bull. Cal. Math. Soc., 75, 185—190 (1983)

AN ALGORITHM FOR NETWORK DECOMPOSITIONS

SUJATA SINHA

(*Received 30 November 1982. Revision Received 7 March 1983*)

Abstract : This paper gives an effective algorithm for decomposition of a given finite sequential machine into a network of smaller machines without feedback.

Introduction : The problem of network decomposition of a sequential machine by using partition pairs has been studied in some detail. But uptil now an algorithm for effecting such a decomposition is not known. This paper gives an efficient step by step algorithm for a network decomposition which may be regarded as an end result in this direction.

1. Notations and basic concepts : An *abstract network* $\mathcal{N}$ in standard form has been defined by Hartmanis and Stearns [1966] to consist of

(*a*) a set of state Machines $M_i = (S_i, I_i, \delta_i)$ $(i = 1, 2, \ldots n)$ called the component machines, where S_i is the set of internal states, I_i the input alphabet and $\delta_i : S_i \times I_i \rightarrow S_i$ the next state function of the *i* th state machine,

(*b*) *I*, a non-empty finite set, called the external input alphabet,

(*c*) 0, a non-empty finite set, called the output alphabet,

(*d*) a set of functions.

$$f_i : \times S_j \times I \rightarrow I_i \qquad (i = 1, 2, \ldots n) \text{ of the form :}$$

$$f_i(s_1, \ldots, s_j, \ldots, s_n, x)$$

$$= (f_{1i}(s_1), \ldots, f_{ji}(s_j), \ldots, f_{ni}(s_n), f_{Ii}(x))$$

where f_{ji} is a function on domain S_j $(j = 1, 2, \ldots n)$ and f_{Ii} a function on domain I. This presupposes that the input letters of the component machine M_i are of the form of $(n+1)$-tuples, of which the element $f_{ji}(s_j)$ will be called the effect of the component M_j in state s_j on the (internal) input of M_i and $f_{Ii}(x)$ the effect of the external input x on M_i, and,

(*e*) a function $g : \times S_i \times I \rightarrow 0$.

The functions f_{ij}, f_{Ii} $(i, j = 1, 2, \ldots n)$ and g will be called the associated functions of the network $\mathcal{N}$ and we may write

$$\mathcal{N} = (M_i, I, 0, f_{ij}, f_{Ii}, g) \quad (i, j, = 1, 2, \ldots, n)$$

The machine correspondong to a given network $\mathfrak{N}$, defined as above, is the Mealy machine $M_{\mathfrak{N}} = (\times S_i, I, 0, \delta, \lambda)$ where the next state function $\delta : \times S_i \times I \to \times S_i$ and the output function $\lambda : \times S_i \times I \to 0$ are given by

$$\delta((s_1, s_2, \dots s_n), x) = \times \delta_i (s_i, f_i((s_1, s_2, \dots s_n), x))$$
$$= \times \delta_i (s_i, (f_{1i}(s_1), f_{2i}(s_2), \dots, f_{ni}(s_n), f_{Ii}(x)))$$

$$\lambda((s_1, s_2, \dots, s_n), x) = g((s_1, s_2, \dots, s_n), x)$$

Any Mealy machine M is said to be realized by or decomposed into a network $\mathfrak{N}$ if M is realized by the corresponding machine $M_{\mathfrak{N}}$.

[A Mealy machine $M=(S, I, 0, \delta, \lambda)$ is said to be realized by another such machine $M'=(S', I, 0, \delta', \lambda')$ if there exists a $1-1$ function α from S into S' such that for every $s \in S$ and $x \in I$

$$\delta'(\alpha(s), x) = \alpha(\delta(s, x)) ; \quad \lambda'(\alpha(s), x) = \lambda(s, x) \text{]}$$

If a machine $M=(S, I, 0, \delta, \lambda)$ is realized by a network

$$\mathfrak{N} = (M_i, I. 0, f_{ij}, f_{Ii}, g) \qquad (i, j = 1, 2, \dots n)$$

then there is a $(1-1)$ function $\alpha : S \to \times S_j$ and we can define partitions τ_i, ρ_{ij} on S and μ_i on $I \; (i, j=1, 2, \dots n)$ as follows:

$$\begin{array}{lll} s \equiv s' \, (\tau_i) & \text{iff (if and only if)} & s_i = s_i' \\ s \equiv s' \, (\rho_{ij}) & \text{iff} & f_{ij}(s_i) = f_{ij}(s_i') \\ x \equiv x' \, (\mu_i) & \text{iff} & f_{Ii}(x) = f_{Ii}(x') \end{array}$$

where

$$\alpha(s) = (s_1, s_2, \dots, s_n), \qquad \alpha(s') = (s_1', s_2', \dots, s_n')$$

and these partiticns satisfy the following set of conditions:

$$\left.\begin{array}{lll} \text{(i)} & \tau_1 \cdot \tau_2 \cdot \dots \tau_n = 0 & \\ \text{(ii)} & \rho_{ij} \geqslant \tau_i & \text{for all } i, j \\ \text{(iii)} & \tau_i \prod_j \rho_{ji} \leqq M_{S-S}(\tau_i) & \text{for all } i, \\ \text{(iv)} & \mu_i \leqslant M_{I-S}(\tau_i) & \text{for all } i, \end{array}\right| \qquad (1.1)$$

Conversely, it has been proved that if for a given machine $M=(S, I, 0, \delta, \lambda)$, a set of partitions $\tau_i, \rho_{ij}, \mu_i \; (i, j=1, 2, \dots, n)$ can be found such that the above conditions are satisfied, then a network with n components can be constructed which would realize M and the construction is as follows:

Set $S_i = \{ B_{\tau_i} \}$, where B_{τ_i} stands for a typical block of the partition τ_i

$$f_{ij}(B_{\tau_i}) = B_{\rho_{ij}}(B_{\tau_i})$$

which denotes that block of ρ_{ij} which contains the block B_{τ_i},

$$f_{I_i}(x) = B_{\mu_i}(x), \qquad I_i = \underset{j}{\times}\{B_{\rho_{ji}}\} \times \{B_{\mu_i}\}$$

$$\delta_i\left(B_{\tau_i}, (\underset{j}{X} B_{\rho_{ji}}(B_{\tau_j}), (B_{\mu_i}(x))\right) = B_{\tau_i}(\delta(s, x)) \text{ if } s \in B_{\tau_i} \cap \bigcap_j B_{\rho_{ji}}(B_{\tau_j}) \qquad (1.2)$$

and the value is arbitrary if $B_{\tau_i} \cap \bigcap_j B_{\rho_{ji}}(B_{\tau_j}) = \phi$ and,

$$\left.\begin{array}{ll} g(\underset{i}{\times} B_{\tau_i}, x) = \lambda(s, x) & \text{if } \bigcap_i B_{\tau_i} = \{s\} \\ \text{and arbitrary if} & \text{if } \bigcap_i B_{\tau_i} = \phi \end{array}\right] \qquad (1.3)$$

2. Algorithm for feed-back-free network decompositions: We now seek to develop a step by step algorithm for choosing the above mentioned partitions τ_i, ρ_{ij}, μ_i for decomposing a given machine into a network without feed back. For simplicity, we shall take in the sequel $\mu_l = 0$ so that the condition (iv) of (1.1) is satisfied.

Algorithm. For decomposing a given finite machine M into a feed-back-free network we proceed as follows:

1. Compute all *Mm* pairs for the machine M.
2. Choose a set of partitions $\tau_1, \tau_2, \ldots, \tau_n$ such that the following conditions are satisfied

$$\left.\begin{array}{ll} \text{(i)} & \tau_1, \tau_2 \ldots \tau_n = 0 \\ \text{(ii)} & \tau_1, \tau_2 \ldots \tau_i \leq M(\tau_i) \quad (i = 1, 2, \ldots, n-1) \end{array}\right] \qquad (2.1)$$

i.e. τ_1 is S.P. $(\tau_1, \tau_2, \tau_2), \ldots \ldots \ldots (\tau_1 \cdot \tau_2 \ldots \tau_{n-1}, \tau_{n-1})$ are partition pairs.

3. For a set of partitions $\tau_1, \tau_2, \ldots, \tau_n$ obtained in step 2, choose partitions ρ_{ij} such that

$$\left.\begin{array}{ll} \text{(i)} & \rho_{ij} \geq \tau_i \quad \text{for} \quad i < j, \quad \rho_{ij} = I \quad \text{for} \quad i \geqslant j \\ \text{(ii)} & \tau_i \prod_{J<i} \rho_{ji} \leq M(\tau_i) \quad (i = 1, 2, \ldots, n-1) \end{array}\right] \qquad (2.2)$$

4. A particular universal choice for ρ_{ij} in step 3 is given by $\rho_{ij} = \tau_i$ for $i < j$, $\rho_{ij} = I$ for $i \geqslant j$.

THEOREM: *The above algorithm is actually effective.*

Proof: The condition of no feed back is characterised by

$$\rho_{ij} = I \qquad \text{for} \qquad i \geq j.$$

Hence by (1.1) (ii)

$$\Pi \rho_{ji} = \prod_{j<i} \rho_{ji} \geq \tau_1, \tau_2, \ldots, \tau_{i-1}$$

so that (1.1) (iii) gives

$$M(\tau_i) \geq \tau_l \,.\, \Pi\, \rho_{ji} \geq \tau_1 \,.\, \tau_2 \cdots \tau_i.$$

This shows (2.1) is a necessary condition. (2 2) follows from (1.1) by using the condition of no-feed back. Corresponding to this, the network $\mathfrak{N}$ is givenby :

$$S_i = \{ B_{\tau_i} \} \qquad f_{ij}\,(B_{\tau_i}) = B_{\tau_i} \quad \text{for} \quad i < j$$

$$f_{ij}\,(B_{\tau_i}) = S \quad \text{for} \quad i \geqslant j$$

$$f_{I_i}(x) = x.$$

$$I_i = \{ (B_{\tau_1}, B_{\tau_2}, \dots \dots, B_{\tau_{i-1}}, S, \dots S, x) \mid x \epsilon I \}$$

$$\delta_I\,(B_{\tau_i}, (B_{\tau_1}, \dots, B_{\tau_{i-1}}, S, \dots, S, x)) = B_{\tau_i}\,(\delta\,(s, x))\ \text{if}\ s \epsilon \bigcap_{j=1}^{i} B_{\tau_i}$$

$$= \text{arbitraty if} \bigcap_{j=1}^{i} B_{\tau_i} = \phi$$

$$(i = 1, 2, \dots n).$$

3. *Example 1* : $M = (S, I, 0, \delta, \lambda)$ where $S = \{ 1, 2, \dots 16 \}$, $I = \{ 0, 1, 2 \}$ $0 = \{ 0, 1 \}$ and δ, λ are given by the table below :

TABLE—I : *Table for* δ, λ

x \ s	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	9,0	14,1	15,0	15,0	12,0	11,1	13,0	11,1	1,1	5.0	6,0	8,1	5,0	2,1	1,0	4,0
1	14,1	9,0	14,1	16,0	10,0	10,0	12,1	10,0	4,0	5,0	7,1	7,1	5,0	3,1	3,0	3,1
2	16,0	15,1	9,0	14,0	12,0	10,1	11,1	13.0	2,1	6,0	8,0	5.0	5,1	1,0	3,0	2,0

By computing all the M_m pairs of this machine (the list of which is long and not reproduced here), we take for τ_1 the S. P. partition :

$$\tau_1 = \{ [1, 2, 3, 4, 5, 6, 7, 8], \quad [9, 10, 11, 12, 13, 14, 15, 16] \}$$

Choose

$$\tau_2 = \{ [1, 2, 3, 4, 10, 11, 12, 13], \quad [5, 6, 7, 8, 9, 14, 15, 16] \}$$

so that

$$\tau_1 \,.\, \tau_2 = \{ [1. 2, 3, 4], \quad [5, 6, 7. 8], \quad [10, 11, 12, 13], \quad [9, 14, 15, 16] \}$$

and $\tau_1 \,.\, \tau_2 \leqslant M(\tau_2)$. Then τ_3 is given by

$$\tau_3 = \{ [1, 5, 9, 10], \quad [2, 6, 11, 14], \quad [3, 7, 12, 15], \quad [4, 8, 13, 16] \}$$

so that $\tau_1 \,.\, \tau_2 \,.\, \tau_3 = 0$. Write

$a_1 = [1, 2, 3, 4, 5, 6, 7, 8]$, $a_2 = [9, 10, 11, 12, 13, 14, 15, 16]$

$b_1 = [1, 2, 3, 4, 10, 11, 12, 13]$, $b_2 = [5, 6, 7, 8, 9, 14, 15, 16]$

$c_1 = [1. 5, 9, 10]$, $c_2 = [2, 6, 11, 14]$, $c_3 = [3, 7, 12, 15]$,

$c_4 = [4, 8, 13, 16]$.

$S_1 = \{ a_1, a_2 \}$; $S_2 = \{ b_1, b_2 \}$; $S_3 = \{ c_1, c_2, c_3, c_4 \}$

$I_1 = \{ x_l \mid l = 0, 1, 2 \}$ where $x_l = (S, S, S\, l)$

$I_2 = \{ y_{il} \mid i = 1, 2 ;\ l = 0, 1, 2, \}$ where $y_{il} = (a_i, S, S, l)$

$I_3 = \{ z_{ijl} \mid i = 1, 2 ;\ j = 1, 2 ;\ l = 0, 1, 2 \}$ where $z_{ijl} = (a_i, b_j, S, l)$

TABLE—II : *Table for* δ_1

$a_i \diagup x_l$	x_0	x_1	x_2
a_1	a_2	a_2	a_2
a_2	a_1	a_1	a_1

TABLE—III : *Table for* δ_2

$b_j \diagup y_{il}$	y_{10}	y_{11}	y_{12}	y_{20}	y_{21}	y_{22}
b_1	b_2	b_2	b_2	b_2	b_2	b_2
b_2	b_1	b_1	b_1	b_1	b_1	b_1

TABLE—IV : *Table for* δ_3

$c_k \diagup z_{ijl}$	z_{110}	z_{111}	z_{112}	z_{120}	z_{121}	z_{122}	z_{210}	z_{211}	z_{212}	z_{220}	z_{221}	z_{222}
c_1	c_1	c_2	c_4	c_3	c_1	c_3	c_1	c_1	c_2	c_1	c_4	c_2
c_2	c_2	c_1	c_3	c_2	c_1	c_1	c_2	c_3	c_4	c_2	c_3	c_1
c_3	c_3	c_2	c_1	c_4	c_2	c_2	c_4	c_2	c_1	c_1	c_3	c_3
c_4	c_3	c_4	c_2	c_2	c_1	c_4	c_1	c_1	c_1	c_4	c_3	c_2

The next state function δ' *and output function* λ' *for the network machine* M_N *are given by the following table :*

Table – V : Table for δ', λ'

$x \diagup (a_i, b_j, c_k)$	(a_1,b_1,c_1)	(a_1,b_2,c_1)	(a_2,b_1,c_1)	(a_2,b_2,c_1)	(a_1,b_1,c_2)	(a_1,b_2,c_2)	(a_2,b_1,c_2)	(a_2,b_2,c_2)	(a_1,b_1,c_3)	(a_1,b_2,c_3)	(a_2,b_1,c_3)	(a_2,b_2,c_3)	(a_1,b_1,c_4)	(a_1,b_2,c_4)	(a_2,b_1,c_4)	(a_2,b_2,c_4)
0	(a_2,b_2,c_1), 0	(a_2,b_1,c_3), 0	(a_1,b_2,c_1), 0	(a_1,b_1,c_1), 1	(a_2,b_2,c_2), 1	(a_2,b_1,c_2), 1	(a_1,b_2,c_2), 0	(a_1,b_1,c_2), 1	(a_2,b_2,c_3), 0	(a_2,b_1,c_4), 0	(a_1,b_2,c_4), 1	(a_1,b_1,c_1), 0	(a_2,b_2,c_3), 0	(a_2,b_1,c_2), 1	(a_1,b_2,c_1), 0	(a_1,b_1,c_4), 0
1	(a_2,b_2,c_2), 1	(a_2,b_1,c_1), 0	(a_1,b_2,c_1), 0	(a_1,b_1,c_4), 0	(a_2,b_2,c_1), 0	(a_2,b_1,c_1), 0	(a_1,b_2,c_3), 1	(a_1,b_1,c_3), 1	(a_2,b_2,c_2), 1	(a_2,b_1,c_1), 1	(a_1,b_2,c_3), 1	(a_1,b_1,c_3), 0	(a_2,b_2,c_4), 0	(a_2,b_1,c_1), 0	(a_1,b_2,c_1), 0	(a_1,b_1,c_3), 1
2	(a_2,b_2,c_4), 0	(a_2,b_1,c_3), 0	(a_1,b_2,c_2), 0	(a_1,b_1,c_2), 1	(a_2,b_2,c_3), 1	(a_2,b_1,c_1), 1	(a_1,b_2,c_4), 0	(a_1,b_1,c_1), 0	(a_2,b_2,c_1), 0	(a_2,b_1,c_2), 1	(a_1,b_2,c_1), 0	(a_1,b_1,c_3), 0	(a_2,b_2,c_2), 0	(a_2,b_1,c_4), 0	(a_1,b_2,c_1), 1	(a_1,b_1,c_2), 0

Here M_N *turns out to be isomorphic to* M.

4. Hardware realization: We shall now make some important remarks on the hardware realization of the network machine $M_{\mathcal{N}}$. Let r_i denote the number of blocks of the partition τ_i $(i = 1, 2, \ldots, n)$ and r the number of external input symbols.

1. For realization of the component M_i we would require at least $-[-\log_2 r_i]$ flip-flop or the like.

2. The number of input leads necessary for M_i is at least

$$-\sum_{j=1}^{i-1}[-\log_2 r_j]-[-\log_2 r]$$

3. The number of leads which combine to compute the output function g of the network is at least

$$-\sum_{j=1}^{n}[-\log_2 r_j]-[-\log_2 r]$$

4. $G_T : \times S_i \times I^T \to O^T$ denote the descriptor function for the network machine $M_{\mathcal{N}}$ over T cycles of computation. i.e. eorresponding to an input tape of length T, so that $G_T(s_1, s_2, \ldots s_n\,;\, x_1, x_2, \ldots, x_T) = (y_1, y_2, \ldots, y_T)$ where $(y_1, y_2, \ldots, y_T)$ is the output tape produced when the machine is in an initial state $(s_1, \ldots s_n)$ and an input tape $(x_1, \ldots x_T)$ is fed into it. Then since the components M_i are state machines and $M_{\mathcal{N}}$ is a Mealy machine,

$$C_{\Omega}(G_T) \leq (T-1)\sum_{i=1}^{n} C_{\Omega}(\delta_i)+T\,C_{\Omega}(g)$$

where C_{Ω} stands for the combinational complexity over a complete basis Ω.

Acknowledgement. The author is indebted to A. Gupta, Reader in Applied Mathematics. Calcutta University for guidance in preparation of this work. The author is grateful to the University Grants Commission for awarding a Research Fellowship. The author also wishes to thank the referee for his valuable suggestions for improvement of the work.

References

Hartmanis, J. and Stearns, R. E. (1966): "Algebraic Structure Theory of Sequential Machines", Prentice-Hall, Inc. Englewood Cliffs, N. J., Chapters—1, 2. 3.

DEPARTMENT OF APPLIED MATHEMATICS,
UNIVERSITY COLLEGE OF SCIENCE,
92, ACHARYA PRAFULLA CHANDRA ROAD,
CALCUTTA-700009.

Bull. Cal. Math. Soc., 75, 191–195 (1983)

ON PSEUDO IDEAL SERIES OF GROUPS

T. K. Dutta

(*Received 29 September 1981*)

Abstract : A nonempty subset A of a semigroup S is called a left pseudo-ideal if (i) $ab \in A$ whenever $a, b \in A$, (ii) $x^2a \in A$ whenever $x \in S$ and $a \in A$. The right pseudo-ideal is defined analogously. A nonempty subset A of a semigroup S is called a pseudo-ideal if it is a left pseudo-ideal as well as a right pseudo-ideal. Example verifies that a group may contain pseudo-ideals. A pseudo-ideal series of a group G is a finite series of subgroups $G = G_0 \supset G_1 \supset \dots \dots \supset G_r$ such that each G_i $(i = 1, \dots, r)$ is a proper pseudo-ideal of G_{i-1} and G_r is pesudo-simple. In this paper we have studied some properties of pseudo-ideal series and obtained criteria for determining the solubility of a group with the help of pseudo-ideal and pseduo-ideal series.

1. Introduction : A nonempty subset A of a semigroup S is called a left pseudo ideal if (i) $ab \in A$ whenever $a, b \in A$ and (ii) $x^2a \in A$ whenever $x \in S$ and $a \in A$ [Sen 1975]. The right pseudo ideal is defined analogously. It is evident from the example of [Sen & Dutta 1979] that a group may contain pseudo-ideals. Now the main purpose of this paper is to define pseudo-ideal series in a group and to determine the criteria for the solubility of a group with the help of pseudo ideal series.

2. We recall the following properties of pseudo-ideals in a group.

2.1 Proposition. *In a group a left pseudo-ideal is a right pseudo-ideal and vice-versa* (Sen 1975).

2.2 Proposition. *Every pseudo-ideal of a group is also a normal subgroup but the converse is not true* (Sen 1975).

2.3 Proposition. *A normal subgroup H of a group G will be a pseudo-ideal if and only if the factor group G/H is of exponent 2* (Sen & Dutta 1979).

2.4 Proposition. *If N is a pseudo-ideal of a group G and A is a pseudo-ideal of G containing N, then A/N is a pseudo-ideal of G/N. Conversely, if N is a pseudo-ideal of G then each pseudo-ideal of G/N has the from A/N for a suitable pseudo-ideal A of G* (Sen & Dutta 1979).

2.5 A semigroup S is called pseudo-simple if it contains no proper pseudo-ideal.

2.6 Proposition. *In a group G a subsemigroup will be a pseudo-ideal if and only if it contains a pseudo-ideal of G* (Sen & Dutta 1979)

Now we prove the following propositions.

2.7 PROPOSITION. *A subgroup A of a group G will be a pseudo-ideal if and only if $xAx \subseteq A$ for all $x \in G$.*

Proof. Let a subgroup A of a group G be also a pseudo-ideal; then A is also a normal subgroup of G (Sen 1975). Hence $xAx = x.xA = x^2A \subseteq A$ for all $x \in G$. Conversely, let A be a subgroup of G and $xAx \subseteq A$ for all $x \in G$. Since 1, the identity element of G belongs to A, $x^2 = x \,.\, 1 \,.\, x \in x\,Ax \subseteq A$. So $x^2A \subseteq A$. Hence A is a pseudo-ideal of G.

2.8. PROPOSITION. *Let S be a semigroup, G be a pseudo-ideal of S and H be a pseudo-simple subsemigroup of S. Now if $G \cap H \neq \phi$ then $H \subseteq G$.*

Proof: Let $a \in G \cap H$. Now for any $x \in H$, $x^2a \in H$ and also $x^2a \in G$ since $a \in G$ and G is a pseudo-ideal of S. So $x^2a \in G \cap H$. Hence $G \cap H$ is a pseudo-ideal of H. But H is pseudo-simple. So $G \cap H = H$ i.e. $H \subseteq G$.

2.9 *Corollary*: *Let G be a group and H be a pseudo-simple subgroup of G; then H is contained in all the pseudo-ideals of G.*

Proof: The corollary follows from the above proposition and the proposition 2.2.

2.10 *Corollary*: In a group G there is a unique pseudo-ideal which is pseudo-simple.

2.11 **Definition**. A pseudo-ideal series of a group G is a finite series of subgroups $G = G_0 \supset G_1 \supset G_2 \supset G_3 \supset \dots \supset G_r$ such that each G_i $(i = 1, \cdots\ , r)$ is a proper pseudo-ideal of G_{i-1} and G_r is pseudo-simple. The length of this series is defined to be the integer r.

2.12 *Example*. Let G be a cyclic group of order 18 generated by an element a. Now $b = a^6 \in G$ and $c = a^2 \in G$. The subgroups G_1 and G_2 generated by b and c respectively are the subgroups of odd order. Hence they contain no proper pseudo-ideals, This shows that a group may contain more than one subgroups which are pseudo-simple.

2·13 PROPOSITION. *Let G_r and H_s be two pseudo-simple subgroups of a Group G such that $G = G_0 \supset G_1 \supset G_2 \supset \dots \supset G_r$ and $G = H_0 \supset H_1 \supset H_2 \supset \dots H_s$ are two pseudo-ideal series of G; then $G_r = H_s$.*

Proof: H_s being a pseudo-simple subgroup of G is contained in G_1 by corollary 2.9. Again, G_2 is a pseudo-ideal of G_1 and H_s is a pseudo-simple subgroup of G_1 so $H_s \subseteq G_2$. Continuing this process we get $H_s \subseteq G_r$ Similarly $G_r \subseteq H_s$. Hence $H_s = G_r$.

2.14 **Definitioa.** If a group G has two pseudo-ideal series then the second onc is called a refinement of the first if every member of the first is also a member of the second.

2.15 **Definition.** Two pseudo-ideal series of a group G, say $G=G_0 \supset G_1 \ldots \ldots \supset G_r$ and $G = H_0 \supset H_1 \supset \ldots \supset H_s = G_r$ are isomorphic if there is a one-to-one correspondence between the set of non-trivial factors G_{i-1}/G_i and set of non-trival factors H_{j-1}/H_j such that the corresponding faetors are isomorphic.

2.16 THEOREM. *Any two pseudo-ideals series of a given group G have isomorphic refinements.*

Proof: Let the two pseudo-ideal series for the group G be those named in the preceding definition. The $G=G_0 \supset G_1 \supset \ldots \supset G_{r+1} = 1$ and $G=H_0 \supset H_1 \supset \ldots \supset H_s = G_r \supset H_{s+1} = 1$ are two normal series of the group G, since every pseudo-ideal of a group is also a normal subgroup. Then, by theorem 6.21 (Macdonald) we have isomorphic refinements of these two normal series by inserting between G_{i-1}, G_i for each $i\,(i=1, \ldots, r+1)$ the following series $G_{i-1}=G_i\,(G_{i-1} \cap H_0) \supset G_i\,(G_{i-1} \cap H_1) \supset G_i\,(G_{i-1} \cap H_2) \supset \ldots \ldots \supset G_i\,(G_{i-1} \cap H_{s+1})$ and between each H_{j-1} and H_j the series $H_{j-1} = H_j\,(G_0 \cap H_{j-1}) \supset H_j\,(G_1 \cap H_{j-1}) \supset \ldots \supset H_j\,(G_{r+1} \cap H_{j-1})$. Now we shall show that $G_i\,(G_{i-1} \cap H_s) = G_i$. Since $H_s=G_r \subset G_{i-1}$. So $H_s \cap G_{i-1} = H_s = G_r$. Now $G_i\,(G_{i-1} \cap G_r) = G_iG_r \subseteq G_iG_i \subseteq G_i$. Also, since $1 \epsilon (G_{i-1} \cap G_r)$, $G_i \subseteq G_i\,(G_{i-1} \cap G_r)$. Thus, $G_i = G_i\,(G_{i-1} \cap G_r) = G_i\,(G_{i-1} \cap H_s)$. Next, we shall show that $G_i\,(G_{i-1} \cap H_j)$ is a pseudo-ideal of $G_i\,(G_{i-1} \cap H_{j-1})$ $(i=1, 2, \cdots \cdots, s)$. $G_i\,(G_{i-1} \cap H_j)$ is a normal subgroup of $G_i\,(G_{i-1} \cap H_{j-1})$. Let $g_ig_{i-1} \epsilon G_i(G_{i-1} \cap H_{j-1})$ where $g_i \epsilon G_i$ and $g_{i-1} \epsilon (G_{i-1} \cap H_{j-1})$. Now $g_ig_{i-1}\,G_i\,g_ig_{i-1} = g_ig_{i-1}\,G_i\,g_{i-1}^{-1}\,g_{i-1}g_lg_{i-1} \subseteq g_i\,G_i\,g_{i-1}\,g_i\,g_{i-1} = g_t\,G_i\,g_{i-1}\,g_i\,g_{i-1}^{-1}\,g_{i-1}^2 = g_i\,G_i\,g_i'\,g_{i-1}^2 = g_i\,G_i\,g_i'' \subseteq G_i$ (since G_i is a pseudo-ideal of G_{i-1}. So by proposition 2.7, G_i is a pseudo-ideal of $G_i\,(G_{i-1} \cap H_{j-1})$. Also, $G_i \subseteq G_t\,(G_{i-1} \cap H_j)$ since $1 \epsilon (G_{i-1} \cap H_{j-1})$. So, by proposition 2.6 $G_i\,(G_{i-1} \cap H_j)$ is a pseudo-ideal of $G_i\,(G_{i-1} \cap H_{j-1})$. Hence, from the above isomorphic refinements of two normal series we get isomorphic refinements of the pseudo-ideal series. Hence the theorem.

2.17 THEOREM. *Let G be a group and N be a pseudo-ideal of G and S be a non pseudo-simple subgroup of G ; then*

(1) if G has a pseudo-ideal series $G=G_0 \supset G_1 \supset \ldots \supset G_r$ and S contains G_r, then S has also a pseudo-ideal series of length r.

(2) if G has a pseudo-ideal series of length r then G/N has also a pseudo-ideal series of length r.

(3) if G/N and N have pseudo-ideal series of length s and t respectively, then G has a pseudo-ideal series of length $s+t$.

(4) if two groups G_1 and G_2 have pseudo-series of length s and t respectively ; then $G_1 \times G_2$ has a pseudo-ideal series of length $s+t$.

Proof: (1) In S we have a pseudo-ideal series $S = S \cap G \supset S \cap G_1 \supset \ldots \supset S \cap G_r = G_r$ from which (1) follows.

(2) By corollary 2.9 $G_r \subseteq N$. Now, let $G \supset N \supset N_1 \supset \cdots \supset G_r$ and $G = G_0 \supset G_1 \supset \ldots \supset G_r$ be two pseudo-ideal series of G.

By Theorem 2.16 these two series have isomorphic refinements. Now by the method of construction of refinements as in theorem 2.16 we have the series $G \supset H_1 \supset \cdots \supset H_{r-1} \supset N$. So we have $G/N \supset H_1/N \supset \ldots \supset H_{r-1}/N$ which shows that G/N has also a pseudo-ideal series of length r.

(3) Let $G/N = G_0/N \supset \cdots \supset G_s/N = N$ and $N = N_0 \supset N_1 \cdots \supset N_{t-1} \supset N_t$ be two pseudo-ideal series of length s and t respectively. Now corresponding to the pseudo-ideals $G_1/N, \ldots G_s/N$ there exist pseudo-ideals $G_1' \cdots \cdots G_s'$ in G, by proposition 2.4, containing N. Thus, we have a pseudo-ideal series $G = G_0 \supset G_1' \supset \ldots \supset G_{s-1}' \supset N \supset \ldots \supset N_t$ which shows that G has a pseudo-ideal series length $s+t$.

(4) The result follows from (3) and the fact that $(G_1 \times G_2)/G_1 \simeq G_2$.

2,18 THEOREM. *A group G containing a pseudo-ideal H is soluble if and only if H is soluble.*

Proof. Let G be a soluble group. Since H is a subgroup of G by the theorem 10.07 (Macdonld), H is also soluble. Conversely, let H be soluble. Since the factor group G/H is of exponent two, it is commutative and hence soluble. Now G/H and H are soluble, so G is also soluble.

With the help of above theorem we now prove the following known result.

2.19 *Corollary. The dihedral group D_n and the quaternion group Q_n are soluble.*

Proof. The dihedral group D_n and the quaternion group Q_n contain a pseudo-ideal I which is a cyclic group (Dutta). Now I being a cyclic group is soluble. Hence D_n and Q_n are soluble.

2.20 A group G containing a pseudo-ideal series $G = G_0 \supset G_1 \supset \ldots \ldots \supset G_r$ is soluble if and only if G_r is soluble.

Proof. Let G be a soluble group. Since G_r is a subgroup of G, G_r is also soluble. Conversely, suppose that G_r is soluble. Now G_r is a pseudo-ideal of G_{r-1}, hence by the preceding theorem G_{r-1} is also soluble. Continuing this process we find that G is also soluble.

I am indebted to Dr. M. K. Sen for his guidance in the preparation of this paper.

References

Dutta, T. K, ( ) : Peeudo-ideals in some finite groups and in finitely generated abelian groups, (Communicated).

Macdonald, I. D. (1968) : The theory of groups, Oxford University Press, Ely House, London W.I.

Sen, M. K. (1975) : On pseudo-ideals of semigroups. *Bull. Cal. Math. Soc.*, 67, 2, 109.

Sen, M. K. and Dutta T. K. (1979) : On pseudo-ideals of groups. Analele Stiintific ale universitatii *'AL. I. CUZA' din I as i Tomul* XXV, S. Ia, f. 2.

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF CALCUTTA,
35, BALLYGUNGE CIRCULAR ROAD,
CALCUTTA-700 019, INDIA.

Bull. Cal. Math. Soc.. 75, 196—204 (1983)

UNSTEADY FLOW OF A DUSTY VISCOUS INCOMPRESSIBLE GAS THROUGH A CIRCULAR TUBE UNDER TIME DEPENDENT PRESSURE GRADIENT

P. MITRA

(*Received 1 August 1979*)

Abstract : The differential equations governing the gas and dust velocities for an unsteady laminar motion of a dusty viscous incompressible gas through a circular tube under any time dependent pressure gradient are obtained. It is shown that the exact solution can be obtained in three cases and in each case, the expression for the velocities of the gas and dust are obtained in closed forms and finally in the case when the pressure gradient decreases exponentially with time, the velocities of the dusty gas, clean gas and dust particles are represented graphically.

1. Introduction : Interest in problems of mechanics of systems with more than one phase has developed rapidly in recent years. Situations which occur frequently are concerned with the motion of a liquid or gas which contains a distribution of solid particles. Such situations arise, for instance the movement of dust laden air, in fluidization, in the use of dust in gas cooling system, in hydrocyclones, in problems of pollution, in tidal waves etc. The mathematical description of such diverse systems must, of course, vary widely. In problems of fluidization, for example, the bulk concentration of particles is large, whereas in problems of dust flow, this will be small. This factor may bring considerable simplifications to the theory of dusty gas flows.

In Saffman's (1962) formulation of the dynamics of a dusty gas, the dust particles are represented by a velocity field, $\vec{v}\,(\vec{\gamma}, t)$, and a number density field, $N(\vec{\gamma}, t)$. It is assumed that the dust particles are uniformly distributed; they are of uniform size and similar shape; the bulk concentration (concentration by volume) of dust is very small, and that the steady Stokes' Law of resistance between the particles and gas applies. However, the mass concentration f of dust can be of order unity by allowing the ratio of the densities of the dust and gas to be large. It is also assumed that the sedimentation may be neglected. The equations of motion give rise to two additional independent parameters due to the presence of the dust, which may be taken as the mass concentration f of dust and a relaxation time τ which measures the rate at which the velocity of a dust particle adjusts to change in the gas velocity and depends upon the size of the individual particles.

Using the formulation of Saffman (1962), several authors gave exact solutions for various problems under different conditions. Michael and Miller (1966) dealt with the plane parallel flow problem of a dusty gas. Michael (1968) also considered the steady motion of a sphere in a dusty gas. Michael and Norey (1968) obtained the solution for the laminar flow of a dusty gas between two rotating cylinders. Dube and Srivastava (1972), Verma and Mathur (1973), Singh (1973), Kishore and Pandey (1977), Gupta and Gupta (1977) studied the problems of flow through circular cylinders under various conditions.

In the present paper we have considered the unsteady laminar motion of a dusty viscous incompressible gas through a circular tube under time dependent pressure gradient. The equations of motion for the gas and dust particles are combined into a single differential equation which is shown to be solvable in three cases and in each case the exact solution is obtained in closed form and finally for the case when the pressure gradient decreases exponentially with time, the changes in the velocity profiles for the dusty gas, clean gas and dust particles with time are shown graphically.

2. Fundamental equations of motion : The equation of motion of a dusty viscous incompressible gas due to Saffman (1962) are the following l

$$\frac{\partial \vec{u}}{\partial t}+(\vec{u}\,.\,\nabla)\vec{u}=-\frac{1}{\rho}\,\text{grad}\,p+\nu\nabla^2\vec{u}+\frac{KN}{\rho}\,(\vec{v}-\vec{u}), \tag{1}$$

$$m\left(\frac{\partial \vec{v}}{\partial t}+\vec{v}\cdot\nabla)\,\vec{v}\right)=K(\vec{u}-\vec{v}), \tag{2}$$

$$\text{div}\ \vec{u}=0, \tag{3}$$

and

$$\frac{\partial N}{\partial t}+\text{div}\,(N\vec{v})=0. \tag{4}$$

In these equations $\vec{u}, \vec{v}$ are the gas and dust velocities respectively, p is the gas pressure, m the mass of a dust particle, and K the Stokes' resistance coefficient which for spherically particles of radius 'd' is $6\pi\mu d$, μ is the gas viscosity, ρ is the gas density and $\nu=\mu/\rho$ is the kinematic coefficient of viscosity of the gas. The time relaxation parameter τ is given from (2) by $\tau=m/K$.

3. Formulation of the present problem : Let 'a' be the radius of the circular tube. We consider the origin on the axis of the cylinder, z-axis along the axis of the cylinder. The motion of the dusty viscous incompressible gas is considered along the axis of the cylinder under the influence of a time dependent pressure gradient. Let the cylindrical polar coordinates (r, θ, z) be chosen. Since the tube is very long, the field variables are independent of z and on account of symmetry, they are independent of θ. Therefore, if we denote the components of velocity of the gas and dust particles in (r, θ, z) directions by (u_1, u_2, u_3) and (v_1, v_2, v_3) respectively, then we have,

$$\left.\begin{aligned} &u_1 = 0, \quad u_2 = 0, \quad u_3 = u_3(\gamma, t) \\ \text{and} \quad &v_1 = 0, \quad v_2 = 0, \quad v_3 = v_3(\gamma, t) \end{aligned}\right\} \tag{5}$$

Let us assume the number density of the dust particles to be constant i.e. $N = N_0$ (constant). Then the equations (3) and (4) reduce to

$$\frac{\partial u_3}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial v_3}{\partial z} = 0.$$

Thus, u_3 and v_3 are independent of z. The equations (1) and (2) reduce respectively to

$$-\frac{1}{\rho}\frac{\partial p}{\partial \gamma} = 0. \tag{6}$$

$$\frac{\partial u_3}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_3}{\partial \gamma^2} + \frac{1}{\gamma}\frac{\partial u_3}{\partial \gamma}\right) + \frac{KN_0}{\rho}(v_3 - u_3) \tag{7}$$

$$m\frac{\partial v_3}{\partial t} = K(u_3 - v_3). \tag{8}$$

Let us now transform the equations (6), (7) and (8) by the use of the following non-dimensional quantities:

$$R = \frac{\gamma}{a}, \quad \bar{z} = \frac{z}{a}, \quad \bar{p} = \frac{pa^2}{\rho\nu^2}, \qquad T = \frac{t\nu}{a^2}, \quad u = \frac{u_3 a}{\nu},$$

$$v = \frac{v_3 a}{\nu}.$$

Then the equations (6), (7) and (8) respectively reduce to

$$\frac{\partial \bar{p}}{\partial R} = 0 \tag{9}$$

$$\frac{\partial u}{\partial T} = -\frac{\partial \bar{p}}{\partial \bar{z}} + \left(\frac{\partial^2 u}{\partial R^2} + \frac{1}{R}\frac{\partial u}{\partial R}\right) + \lambda_1(v - u) \tag{10}$$

and

$$\lambda_2\frac{\partial v}{\partial T} = u - v. \tag{11}$$

where $\lambda_1 = f/\lambda_2 = N_0Ka^2/\rho\nu$, $f = N_0 m/\rho$ = the mass concentration of dust particles, $\lambda_2 = \nu m/Ka^2$. From (9) and (10) it follows that $\partial\bar{p}/\partial\bar{z}$ is a function of T and let it be $\psi(T)$. Thus,

$$-\frac{\partial \bar{p}}{\partial \bar{z}} = \psi(T) \tag{12}$$

Then from (10) we get

$$\frac{\partial u}{\partial T} = \psi(T) + \left(\frac{\partial^2 u}{\partial R^2} + \frac{1}{R}\frac{\partial u}{\partial R}\right) + \lambda_1(v - u). \tag{13}$$

Eliminating v from (11) and (13) we get,

$$\frac{\partial^2 u}{\partial T^2}+\left(\lambda_1+\frac{1}{\lambda_2}\right)\frac{\partial u}{\partial T}-\frac{\partial}{\partial T}\left(\frac{\partial^2 u}{\partial R^2}+\frac{1}{R}\,\frac{\partial u}{\partial R}\right)-\frac{1}{\lambda_2}\left(\frac{\partial^2 u}{\partial R^2}+\frac{1}{R}\,\frac{\partial u}{\partial R}\right)$$

$$-\left[\,\psi'(T)+\frac{1}{\lambda_2}\,\psi(T)\right]=0, \tag{14}$$

where dash in $\psi(T)$ denotes differentiation with respect to T.

Let us choose the solution of (14) as

$$u=F(R)\,.\,\psi(T). \tag{15}$$

The boundary conditions are :

$$\left.\begin{aligned} u&=0 &&\text{at}\quad R=1,\\ u&=\text{finite} &&\text{at}\quad R=0,\\ \text{i.e.}\;\; F(R)&=0 &&\text{at}\quad R=1,\\ F(R)&=\text{finite} &&\text{at}\quad R=0. \end{aligned}\right\} \tag{16}$$

Then the equation (14) with the help of (15) reduces to

$$\frac{d^2F}{dR^2}+\frac{1}{R}\,\frac{dF}{dR}+\beta\left(F+\frac{1}{\beta}\right)=0, \tag{17}$$

where

$$\beta=-\frac{\psi''(T)+(\lambda_1+1/\lambda_2)\,\psi'(T)}{\psi'(T)+(1/\lambda_2)\psi(T)} \tag{18}$$

i.e.

$$\psi''(T)+[(\lambda_1+1/\lambda_2)+\beta]\,\psi'(T)+(\beta/\lambda_2)\,\psi(T)=0, \tag{19}$$

β being a constant. Equation (17) is the differential equation which determines F under the conditions (16) and then from (15) and (10), u and v can be determined, provided the form of the function $\psi(T)$ is known.

Let us now obtain the solution of (17) in the following cases :—

Case (*i*). Let $\beta>0$.

If $(\lambda_1+(1/\lambda_2)+\beta)^2>4\,.\,\beta/\lambda_2$, then both the roots of the auxiliary equation of (19) are real and negative. So we can choose

$$\psi(T)=\alpha\,.\,e^{-\lambda^2 T} \tag{20}$$

where α and λ are real constants. Then the equation (17) reduces to

$$\frac{d^2F}{dR^2}+\frac{1}{R}\,\frac{dF}{dR}+h^2\left(F+\frac{1}{h^2}\right)=0, \tag{21}$$

where

$$h^2=\frac{\lambda^2\,(1+\lambda_1\,\lambda_2-\lambda_2\,\lambda^2)}{1-\lambda_2\,\lambda^2}\,. \tag{22}$$

The solution of the differential equation (21) is

$$F(R)=A_1\,J_0(hR)+B_1\,Y_0(hR)-\frac{1}{h^2}\,, \tag{23}$$

where A_1 and B_1 are constants, J_0 and Y_0 are Bessel's functions of the first and second kind of order zero respectively. Applying the boundary conditions (16), we get from (23),

$$A_1 = \frac{1}{h^2} \cdot \frac{1}{J_0(h)} \quad \text{and} \quad B_1 = 0.$$

From (23) we get, $F(R) = (1/h^2)\left[\{J_0(hR)/J_0(h)\} - 1\right]$. Hence, form (15), the gas velocity is given by

$$u = \frac{\alpha}{h^2}\left[\frac{J_0(hR)}{J_0(h)} - 1\right] . e^{-\lambda^2 T} \tag{24}$$

Substituting this value of u in (13) we get after simplification

$$v = \frac{\alpha(\lambda_1^2 - \lambda^2 + h^2)}{h^2 \lambda_1}\left[\frac{J_0(hR)}{J_0(h)} - 1\right]. \; e^{-\lambda^2 T} \tag{25}$$

This gives the dust velocity.

Case (*ii*): Let $\beta > 0$.

If $(\lambda_1 + (1/\lambda_2) + \beta)^2 < 4 . \beta/\lambda_2$, then the roots of the auxiliary equation of (19) are imaginary. Thus we can take

$$\psi(T) = \mu_1 \, e^{(\alpha_1 + i\beta_1) T}, \tag{26}$$

where μ_1, α_1 and β_1 are real constants. Thus, the equation (17) simplifies to

$$\frac{d^2F}{dR^2} + \frac{1}{R}\frac{dF}{dR} - p^2\left(F - \frac{1}{p^2}\right) = 0, \tag{27}$$

where

$$p^2 = \frac{(\alpha_1 + i\beta_1)^2 + (\lambda_1 + 1/\lambda_2)(\alpha_1 + i\beta_1)}{(\alpha_1 + i\beta_1) + 1/\lambda_2}. \tag{28}$$

We now put $F - 1/p^2 = y$. Then (27) reduces to

$$\frac{d^2y}{dR^2} + \frac{1}{R}\frac{dy}{dR} - p^2 y = 0. \tag{29}$$

The solution of (29) is $y = A_2 I_0(PR) + B_2 K_0(PR)$

$$\text{i.e.} \quad F(R) = A_2 I_0(PR) + B_2 K_0(PR) + \frac{1}{p^2}. \tag{30}$$

where I_0 and K_0 are modified Bessel functions of first and second kind of order zero respectively. The solution of equation (30) under the boundary conditions (16) is $F(R) = (1/p^2)\left[1 - \{I_0(PR)/I_0(P)\}\right]$ and so from (15), the gas velocity is given by

$$u = \frac{\mu_1}{p^2}\left[1 - \frac{I_0(PR)}{I_0(P)}\right] e^{(\alpha_1 + i\beta_1) T}. \tag{31}$$

Putting this value of u in (13) we get after due simplification

$$v = \frac{\mu_l(\lambda_1 + \alpha_1 + i\beta_1 - p^2)}{\lambda_1 p^2}\left[1 - \frac{I_0(PR)}{I_0(P)}\right] e^{(\alpha_1 + i\beta_1)T} \tag{32}$$

This gives the velocity of the dust particle. In equations (31) and (32) the real parts of u and v have only the physical significance.

Case (iii): Let $\beta < 0$.

In this case, the roots of the auxiliary equation of (19) are real, one is negative and the other is positive. Here, the velocities of the gas and dust particles can be obtained as in case (i).

4. Numerical Calculations (Pressure gradient decreasing exponentially with time). The results of numerical calculations (Milton and Stegun 1970; Gray and Mathews 1952) for the velocities of dusty gas, clean gas ($f = 0$) and dust particles for different values of $T (0 \leqslant T \leqslant 1)$ and $R (0 \leq R \leq 1)$ for the case when the pressure gradient decreases exponentially with time are shown in the table.

The graphs of velocity profiles for the dusty gas, clean gas and dust particles are shown in figs. 1, 2 and 3 respectively for different values of R and T.

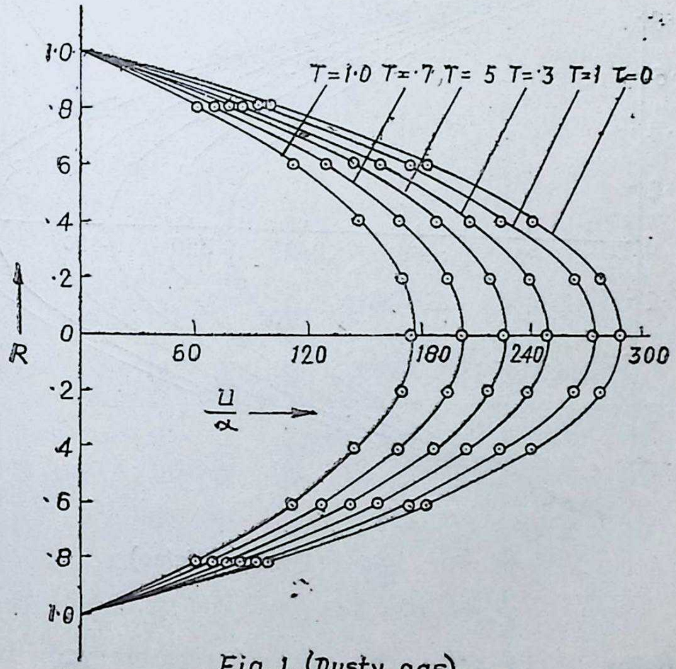


Fig. 1. (Dusty gas)

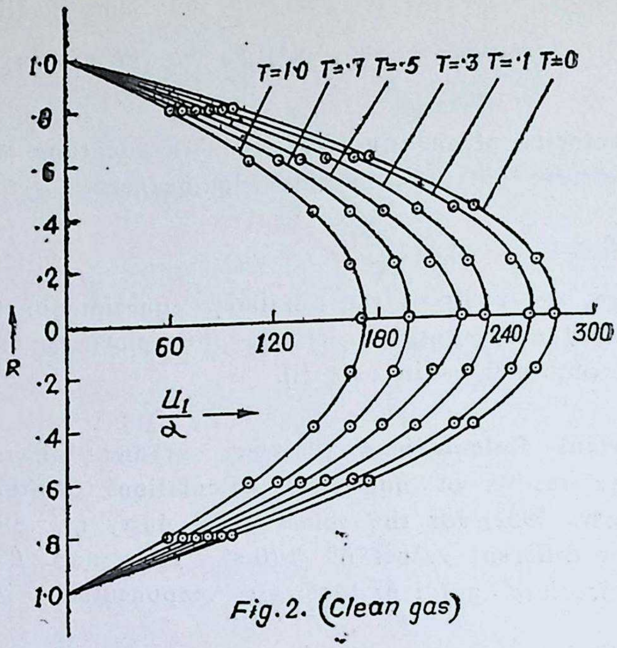


Fig. 2. (Clean gas)

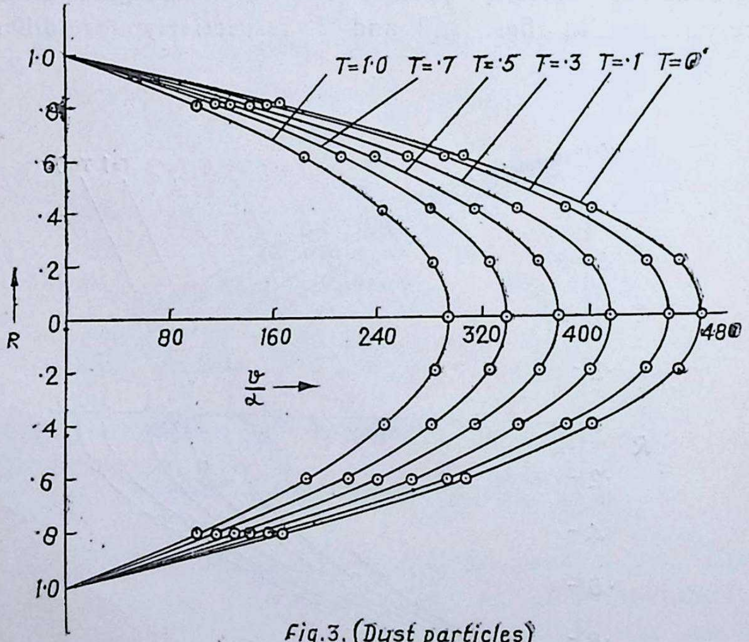


Fig. 3. (Dust particles)

5. Conclusion: The following points are observed from the table and the graphs :—

TABLE

$(f = \cdot 2,\ \lambda_s = \cdot 8,\ \lambda^s = \cdot 5)$

	Gas Velocity						Clean Gas Velocity						Dust Velocity					
	T						T						T					
R	.0	.1	.3	.5	.7	1.0	0	.1	.3	.5	.7	1.0	.0	.1	.3	.5	.7	1,0
0	.2880	.2736	.2477	.2247	.2016	.1757	.2780	.2641	.2390	.2168	,1946	.1696	.4838	.4596	.4160	.3775	.3387	.2951
,2	.2775	.2636	.2387	.2165	,1942	.1693	.2680	.2546	.2305	.2090	,1876	.1635	.4662	.4429	,4009	.3637	.3263	,2843
,4	,2400	.2230	.2064	.1872	,1680	·1464	,2340	,2223	.2012	.1825	,1638	,1427	.4032	,3830	.3467	.3146	,2822	.2459
,6	.1830	.1739	.1574	,1427	.1281	,1116	.1740	.1653	,1496	.1357	.1218	.1061	.3074	.2921	,2644	,2399	.2152	.1875
.8	0990	,0941	.0851	.0772	.0693	.0604	0960	.0912	.0826	.0749	.0672	.0586	.1663	,1580	.1430	.1298	.1164	.1014
1 0	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

(*i*) The velocity of dust particles at any point within the prescribed values of time and distance is greater than that of gas in the present set up, whereas Dube and Srivastave (1972) showed that the velocity of gas is more than that of dust when the pressure gradient varies linearly with time.

(*ii*) When the dust is very fine, τ decreases and when $\tau \rightarrow 0$, the velocity of dusty gas becomes nearly equal to that of clean gas.

(*iii*) It is clear that the gas and dust velocities decrease with T for fixed R and for any T, the gas and dust velocities decrease with the increase of R and are maximum when $R = 0$. From this it follows that both the gas and the dust particles which are nearer to the axis of the tube move with greater velocity.

(*iv*) If the masses of the dust particles are small, then their influence on the fluid flow is reduced, and in the limit as $m \rightarrow 0$. the gas behave as ordinary viscous and we get the laminar flow of a viscous gas in a circular tube under the influence of exponential time dependent pressure gradient.

Acknowledgement : The author expresses his gratefulness to Dr. Ambarish Ghosh, Professor at Indian Statistical Institute, Calcutta and part-time lecturer in the Department of Applied Mathematics, Calcutta University, for his kind supervision. The auther is also thankful to U.G.C. authorities for providing financial assistance to continue the research work.

References

Dube, S. N. and Srivastava, L. P. (1972): *Def. Sc. J.*, 22.

Gray and Mathews (1952): A treatise on Bessel Functions and their applications to Physics, Macmillan and Co. Limited, London.

Gupta, R. K. and Gupta, S. C, (1977): *Proc. Indian National Science Acad.* Part A, 1. 43.

Kishore, Newal and Pandey, R. D. (1977): *Proc. Indian. Acad. Sci.* 85A. 5.

Michael, D, H. and Miller. D. A. (1966): *Mathematika*, 13.

Michael, D. H. (1968): *J. Fluid. Mech.* 31.

Michael, D. H. and Norey, P. W. (1968): *Qua. J. Mech. and App. Math.*, 21.

Milton Abramowitz and Irene A. Stegun (1970): Handbook of Mathematical Functions, Dover Publication, Inc. New York.

Saffman, P. G. (1962): *J. Fluid Mech.* 13.

Singh, Devi (1973): *Indian J. Phys.* 47.

Verma, P. D. and Mathur, A. K: (1973): *Indian Jurnal of Pure and Applied Math* 4.

TEACHER FELLOW,
DEPARTMENT OF APPLIED MATHEMATICS,
CALCUTTA UNIVERSITY, WEST BENGAL.
Present Address :
SREE CHAITANYA COLLEGE,
HABRA, DIST. 24-PARGANAS,
WEST BENGAL.

Bull. Cal. Math. Soc., 75, 205—217 (1983)

DECAY OF EDDY TURBULENCE OBTAINED THROUGH LINEARIZATION OF DYNAMICAL EQUATIONS

BHULU RANJAN SAHA

(*Received 19 June 1979*)

Abstract : Hydrodynamical equations, relating to any fluid flow problem are non-linear in character. Relevant momentum-conservation equations are usually termed as Navier-Stokes equations. In the study of turbulence, we assume that the fluctuating components of connected field variables obey these equations. Accordingly, when we adopt usual averaging technique, we come across an equation with two unknowns at the expense of all the relevant conservation equations. The resulting equation referred to above is known as Karman-Howarth equation in case of homogeneous and isotropic turbulence. In what follows. we shall simplify the relevant dynamical equations with the introduction of a suitable auxiliary function. Next we shall rewrite velocity correlations in terms of expectation values relating to such auxiliary functions and we recast the dynamical equations accordingly. Then using the general rule for merger of neighbouring points, we obtain the associated decay law [Ghosh 1972].

2.1. Introduction : In hydrodynamical problems there have been abortive attempts in linearizing the entire Navier-Stokes equations. But there are occasions when some simplified versions of these equations could be linearized. One such instance is due to J. D. Cole (1951). In our analysis given here, we shall proceed in the line envisazed above, starting from an equation of the type considered by J. D. Cole (1951). We shall show that in case of homogeneous and isotropic turbulence, the energy decays according to the simple law,

$$\bar{u}^2 \propto t^{-1}. \tag{2.1}$$

2.2. Equation of Motion : In the context narrated above, we start with Navier-Stokes equations in somewhat general form as

$$\frac{\partial \vec{q}}{\partial t}+(\vec{q}\,.\,\vec{\nabla})\,\vec{q}=-\frac{1}{\rho}\nabla\,p+\nu\nabla^2\,\vec{q}, \tag{2.2}$$

where $\vec{q}$ is the fluctuating velocity vector at (x, y, z), p is the fluctuating pressure at the same point at a time t, while other symbols have got their usual meanings.

In continuation, we consider the equation of continuity in the form

$$\text{div}\ \vec{q}=0. \tag{2.3}$$

relevant to flow pertaining to an incompressible fluid.

If we take the divergence of the whole of the relation given by (2.2), we obtain

$$\nabla \cdot [(\vec{q} \cdot \vec{\nabla})\vec{q}] = -\frac{1}{\rho}\nabla^2 p. \tag{2.4}$$

From (2.4), we find that the pressure term qualitatively matches the inertia term. Hence for a qualitiative analysis, we can approximate the Navier-Stokes equations in the form

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \vec{\nabla})\vec{q} = \nu \nabla^2 \vec{q}. \tag{2.5}$$

2.3 Linearisation of the Equations of Motion given by (2.5) : Let us follow J. D. Cole (1951) and use an auxiliary function $\theta(x, y, z, t)$ to put

$$\vec{q}\,(x, y, z.\, t) = -2\nu\, \nabla_{x, y, z} \log \theta. \tag{2.6}$$

Substitution of (2.6) in (2.5) gives rise to the equation

$$\left(\vec{\nabla} - \frac{1}{\theta}\vec{\nabla}\theta\right)\left(\frac{\partial \theta}{\partial t} - \nu \nabla^2 \theta\right) = 0. \tag{2.7}$$

The set of solutions of (2.7) is obtainable from

$$\frac{\partial}{\partial t}\theta(x, y, z, t) - \nu \nabla^2 \theta(x, y, z, t) = \psi(t)\, \theta(x, y, z, t), \tag{2.8}$$

where $\psi(t)$ is an arbitrary function of t only.

The solution of the equation (2.8) is obtained as

$$\theta(x, y, z, t) = \frac{e^{c(t)}}{(4\pi\nu t)^{3/2}} \iiint_{-\infty}^{\infty} \theta(X, Y, Z, 0)\, e^{-[(X-x)^2 + (Y-y)^2 + (Z-z)^2]/4\nu t}\, dX \,.\, dY \,.\, dZ \tag{2.9}$$

[Dutta Gupta 1975, P. 117]

where $\theta(X, Y, Z, 0)$ is the value of θ at $(X, Y, Z,$ for $t=0$ and

$$c(t) = \int_0^t \psi(\tau)\, d\tau. \qquad \text{[Sneddon 1951, p. 196].}$$

Now, we are in a position to rewrite

(i) $$\vec{q}\,(x', y', z', t) = -2\nu\, \nabla'_{x', y', z'} \log \theta \tag{2.10}$$

(ii) $$\theta\,(x', y', z', t) = \frac{e^{c(t)}}{(4\pi\nu t)^{3/2}} \iiint_{-\infty}^{\infty} \theta\,(X', Y', Z'\, \theta) \times e^{-[(X'-x')^2 + (Y'-y')^2 + (Z'-z')^2/4\nu t}\, dX' \,.\, dY' \,.\, dZ'. \tag{2.11}$$

where $x' = x + \xi,\ y' = y + \eta,\ z' = z + \rho.$

In estimation, we find

$$\overline{\{\vec{q}(x, y, z, t) \cdot \vec{q}(x', y', z', t)\}}$$

$$= 4\nu^2 \left(\left\{ \frac{\overline{\nabla_{x, y, z}\, \theta(x, y, z, t)} \cdot \overline{\nabla'_{x', y', z'}\, \theta(x', y', z', t)}}{\theta(x, y, z, t) \cdot \theta(x', y', z', t)} \right\} \right) \tag{2.12}$$

Now the equation (2.5) can be expressed as

$$\frac{\partial\vec{q}/\partial t}{\nu\nabla^2\vec{q}} + \frac{(\vec{q} \cdot \vec{\nabla})\,\vec{q}}{\nu\nabla^2\,\vec{q}} = 1$$

i.e.,

$$\frac{\partial\vec{q}/qt}{\nu\nabla^2\vec{q}} + R = 1 \tag{2.13}$$

where $R =$ Reynolds number

$$= \frac{(\vec{q} \cdot \vec{\nabla})\,\vec{q}}{\nu\nabla^2\,\vec{q}} = \frac{\text{Inertia effect}}{\text{Viscous dissipation}}.$$

In the problem under consideration, we assume that the Reynolds number remains unchanged.

Now we note that substitution of

$$q = -2\nu\, \nabla_{x, y, z} \log\theta$$

in (2.13) has given rise to the equation (2.8).

Since at a later stage, we shall have to consider the reciprocal field of $\theta(x, y, z, t)$, we now make a substitution $\chi(x, y, z, t) = 1/\theta(x, y, z, t)$. Hence,

$$\vec{q} = -2\nu\nabla_{x, y, z} \log\theta = -2\nu\nabla_{x, y, z} \log(1/\chi)$$

$$= 2\nu\nabla_{x, y, z} \log\chi. \tag{2.14}$$

Substitution of (2.14) in (2.13) gives

$$(\partial/\partial t)\,\chi(x, y, z, t) - \nu\nabla^2\,\chi(x, y, z, t) = \phi(t)\,\chi(x, y, z, t). \tag{2.15}$$

For the sake of convenience and simplification, let us put $\varphi(t) = -\psi(t)$, so that (2.15) takes the form

$$\frac{\partial}{\partial t}\,\chi(x, y, z, t) - \nu\nabla^2\,\chi(x, y, z, t) = -\psi(t)\,\chi(x, y, t) \tag{2.16}$$

The solution of the equation (2.16) is obtained as

$$\chi(x, y, z, t) = \frac{e^{-\sigma(t)}}{(4\pi\nu t)^{3/2}} \int\!\!\int_{-\infty}^{\infty}\!\!\int \chi(X, Y, Z, 0)\, e^{-[(X-x)^2 + (Y-y)^2 + (Z'-z)^2]/4\nu t}\, dX.\, dY.\, dZ. \tag{2.17}$$

Similarly,

$$\chi(x', y', z', t)=\frac{e^{-c(t)}}{(4\pi\nu t)^{3/2}}\int\!\!\int\!\!\int_{-\infty}^{\infty}\chi(X', Y', Z', 0)\, e^{-[X'-x')^2+(Y'-y')^2+(Z'-x')^2]/4\nu t}\, dX'.\,dY'.\,dZ' \qquad (2.18)$$

Also

$$\overline{\vec{q}\,(x, y, z, t).\,\vec{q}\,(x', y', z', t)}$$

$$4\nu^2=\left(\overline{\frac{\nabla_{x,y,z}\,\chi(x, y, z, t).\,\nabla'_{x',y',z'}\,\chi(x', y', z', t)}{\chi(x, y, z, t)\,\chi(x', y', z', t)}}\right) \qquad (2.19)$$

2.4 Equation of continuity: The equation of continuity is given by

$$\text{div}\ \vec{q}\,(x, y, z, t)=0$$

i.e.,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (2.20)$$

From (2.6) and (2.14), we obtain two sets of equations as

$$\theta\,(\theta_{xx}+\theta_{yy}+\theta_{zz})-(\theta_x^2+\theta_y^2+\theta_z^2)=0 \qquad (2.21)$$

and

$$\chi\,(\chi_{xx}+\chi_{yy}+\chi_{zz})-(\chi_x^2+\chi_y^2+\chi_z^2)=0. \qquad (2.22)$$

Averaging equations (2.21) & (2.22), we obtain

$$\nabla^2_{\xi,\eta,\rho}\,\overline{\theta\,(x, y, z, t)\,\theta\,(x+\xi, y+\eta, z+\rho, t)}=0 \qquad (2.23)$$

and

$$\nabla^2_{\xi,\eta,\rho}\,\overline{\chi\,(x, y, z, t)\,\chi\,(x+\xi, y+\eta, z+\rho, t)}=0 \qquad (2.24)$$

respectively.

As u, v, w, u', v', w' are fluctuating velocities, we have

$$\bar{u}=\bar{u}'=\bar{v}=\bar{v}'=\bar{w}=\bar{w}'=0$$

i.e.

$$\begin{aligned}\overline{\chi\,(x, y, z, t)\,\theta_x\,(x, y, z, t)} &= \overline{\chi\,(x+\xi, y+\eta, z+\rho, t)\,\theta_{x'}\,(x+\xi, y+\eta, z+\rho, t)}\\ =\overline{\chi\,(x, y, z, t)\,\theta_y\,(x, y, z, t)} &= \overline{\chi\,(x+\xi, y+\eta, z+\rho, t)\,\theta_{y'}\,(x+\xi, y+\eta, z+\rho, t)}\\ =\overline{\chi\,(x, y, z, t)\,\theta_z\,(x, y, z, t)} &= \overline{\chi\,(x+\xi, y+\eta, z+\rho, t)\,\theta_{z'}\,(x+\xi, y+\eta, z+\rho, t)}\\ =0, \quad \ldots \quad \ldots \quad \ldots & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2.25)\end{aligned}$$

2.5. Use of Millionshtchikov's hypothesis to obtain the relevant decay law: Assuming the quasinormality hypothesis due to Millionshtchikov, we obtain

$$\overline{(\nabla\theta.\,\nabla'\theta')\,\chi\chi'}=\overline{\chi.\,\nabla\theta}\,.\,\overline{\chi'.\,\nabla'\theta'}+\overline{\nabla\theta.\,\nabla'\theta'}\,.\,\overline{\chi\chi'}+\overline{\chi.\,\nabla'\theta'}\,.\,\overline{\chi'.\nabla\theta} \qquad (2.26)$$

From (2.25) & (2.26) we obtain

$$\overline{u\,(x,y,z,t)\,u\,(x+\xi,y+\eta,z+\rho,t)}$$
$$=4\nu^2\,[\overline{\theta_x\,(x,y,z,t)\cdot\chi\,(x,y,z,t)\,\theta_{x'}\,(x',y',x',t)\,\chi\,(x',y',x',t)}]$$
$$=4\nu^2\,[\overline{\theta_x\,(x,y,z,t)\,\theta_{x'}\,(x+\xi,y+\eta,z+\rho,t)}\cdot\overline{\chi\,(x,y,z,t)\,\chi\,(x+\xi,y+\eta,z+\rho,t)}$$
$$+\overline{\chi\,(x,y,z,t)\,\theta_{x'}\,(x+\xi,y+\eta,z+\rho,t)}\cdot\overline{\chi\,(x+\xi,y+\eta,z+\rho,t)\,\theta_x\,(x,y,z,t)}] \quad (2.27)$$

where $x'=x+\xi,\quad y'=y+\eta,\quad z'=z+\rho,$

Similarly,

$$\overline{v\,(x,y,z,t)\,v\,(x+\xi,y+\eta,z+\rho,t)}$$
$$=4\nu^2\,[\overline{\theta_y\,(x,y,z,t)\,\chi\,(x,y,z,t)\,\theta_{y'}\,(x',y',z',t)\,\chi\,(x',y',z',t)}]$$
$$=4\nu^2\,[\overline{\theta_y\,(x,y,z,t)\,\theta_{y'}\,(x+\xi,y+\eta,z+\rho,t)}\cdot\overline{\chi\,(x,y,z,t)\,\chi\,(x+\xi\;\;y+\eta,z+\rho,t)}$$
$$+\overline{\chi\,(x,y,z,t)\,\theta_{y'}\,(x+\xi,y+\eta,z+\rho,t)}\cdot\overline{\chi\,(x+\xi,y+\eta,z+\rho,t)\,\theta_y\,(x,y,z,t)}] \quad (2.28)$$

and

$$\overline{w\,(x,y,z,t)\,w\,(x+\xi,y+\eta,z+\rho,t)}$$
$$=4\nu^2\,[\overline{\theta_z\,(x,y,z,t)\,\chi\,(x,y,z,t)\,\theta_{z'}\,(x',y'.z',t)\,\chi\,(x',y',z',t)}$$
$$=4\nu^2\,[\overline{\theta_z\,(x,y,z,t)\,\theta_{z'}\,(x+\xi,y+\eta,z+\rho,t)}\cdot\overline{\chi\,(x,y,z,t)\,\chi\,(x+\xi,y+\eta,z+\rho,t)}$$
$$+\overline{\chi\,(x,y,z,t)\,\theta_{z'}\,(x+\xi,y+\eta,z+\rho,t)}\cdot\overline{\chi\,(x+\xi,y+\eta,z+\rho,t)\,\theta_z\,(x,y,z,t)}] \quad (2.29)$$

Since the turbulent field is assumed homogeneous and isotropic, we have

$$[\overline{\theta_{x'}\,(x+\xi,y+\eta,z+\rho,t)\,\chi\,(x,y,z,t)}]_{\xi=\eta=\rho=0}$$
$$=[\overline{\theta_x\,(x,y,z,t)\,\chi\,(x+\xi,y+\eta,z+\rho,t)}]_{\xi=\eta=\rho=0}$$
$$=[\overline{\theta_{x'}\,(x+\xi,y+\eta,z+\rho,t)\,\chi\,(x+\xi,y+\eta,z+\rho,t)}]_{\xi=\eta=\rho=0}$$
$$=\overline{\theta_x\,(x,y,z,t)\,\chi\,(x,y,z,t)}=0 \quad (2.30)$$

and

$$[\overline{\theta\,(x+\xi,y+\eta,z+\rho,t)\,\chi\,(x,y,z,t)}]_{\xi=\eta=\rho=0}$$
$$=[\overline{\theta\,(x,y,z,t)\,\chi\,(x+\xi,y+\eta,z+\rho,t)}]_{\xi=\eta=\rho=0}$$
$$=[\overline{\theta\,(x,+\xi,y+\eta,z+\rho,t)\,\chi\,(x+\xi,y+\eta,z+\rho,t)}]_{\xi=\eta=\rho=0}$$
$$=\overline{\theta\,(x,y,z,t)\,\chi\,(x,y,z,t)}=1 \quad (2.31)$$

Next we calculate

$$\overline{\theta\,(x+\xi,y+\eta,z+\rho,t)\,\chi\,(x,y,z,t)}\;,\;\overline{\chi\,(x,y,z,t)\,\chi\,(x+\xi,y+\eta,z+\rho,t)},$$
$$\overline{\theta_x\,(x,y,z,t)\,\theta_{x'}\,(x+\xi,y+\eta,z+\rho,t)}\;,\;\overline{\chi\,(x,y,z,t)\,\theta_{x'}\,(x+\xi,y+\eta,z+\rho,t)}$$

and

$$\overline{\chi\,(x+\xi,y+\eta,z+\rho,t)\,\theta_{x},\,(x,y,z,t)}$$

for use in (2.27).

(a) Let

$$L(\xi, \eta, \rho, t) = \overline{\theta(x+\xi, y+\eta, z+\rho, t)}\ \overline{\chi(x, y, z, t)}$$

$$= \underset{\substack{\alpha \to \infty \\ \beta \to \infty \\ \gamma \to \infty}}{Lt} \frac{1}{8\alpha\beta\nu} \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\gamma}^{\gamma} \theta(x+\xi, y+\eta, z+\rho, t)\, \chi(x, y, z, t)\, dx\, dy\, dr \tag{2.32}$$

The integration connected with the x-direction is

$$\underset{\alpha \to \infty}{Lt} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \chi(x, t)\, \theta(x+\xi, t)\, dx$$

$$= \underset{\alpha \to \infty}{Lt} \frac{1}{2\alpha} \int\int_{-\alpha}^{\alpha}\int \frac{1}{4\pi\nu t}\, \chi(X, 0)\, \theta(X', 0)\, e^{-[(X-x)^2+(X'-x-\xi)^2]/4\nu t}\, dX\, dX'\, dx$$

$$= \underset{\alpha \to \infty}{Lt} \frac{1}{2\alpha} \int\int_{-\alpha}^{\alpha}\int \frac{\chi(X, 0)\, \theta(X', 0)}{4\pi\nu t}\, e^{-(2x-X-X'+\xi)^2/8\nu t}.$$

$$e^{-(X-X'+\xi)^2/8\nu t}\, dX\, dX'\, dx$$

$$= \underset{\alpha \to \infty}{Lt} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{x(X, 0)\, \theta(X', 0)}{2\sqrt{(2\pi\nu t)}}\, e^{-(X-X'+\xi)^2/8\nu t}\, dX.\, dX'$$

$$= \underset{\alpha \to \infty}{Lt} \frac{1}{2\alpha} \int_{-\alpha}^{\alpha}\int_{-\alpha-X}^{\alpha-X} \frac{\chi(X, 0)\, \theta(X+a, 0)}{2\sqrt{(2\pi\nu t)}}\, e^{-(\xi-a)^2/8\nu t}\, dX.\, da$$

where $X' = X + a$.

$$= \frac{1}{2\sqrt{(2\pi\nu t)}} \int_{-\infty-X}^{\infty-X} \overline{\chi(X, 0)\, \theta(X+a, 0)}\, e^{-(\xi-a)^2/8\nu t}\, da \tag{2.33}$$

The other two sets of integrations give similar results. Then we are in a position to obtain

$$L(\xi, \eta, \rho, t)$$

$$= \frac{1}{(8\pi\nu t)^{3/2}} \int\int_{-\infty}^{\infty}\int L(a, b, a, 0)\, e^{-(\xi-a)^2+(\eta-b)^2+(\rho-c)/8\nu t}\, da.\, db.\, dc \tag{2.34}$$

Let νt be large compared with the values of $a^2+b^2+c^2=s^2$, say, for which $L(a, b, c, 0)$ is assumed to be significant enough to contribute to the integral. This estimation gives rise to the value "1" for the exponential part in the integrand. Now the fact that the correlation coefficient at time t is other than unity leads to the calculation of $\xi^2+\eta^2+\rho^2=r^2$ as of order νt. Naturally, $2(a\xi+b\eta+c\rho)/8\nu t$ is of order $(\nu t)^{-1/2}$, and $s^2/\nu t$ is of order $(\nu t)^{-1}$ as a consequence. (2.34) can now be expressed as

$$L(\xi,\eta,\rho,t)=\frac{e^{-r^2/8\nu t}}{(8\pi\nu t)^{5/2}}\int\int_{-\infty}^{\infty}\int L(a,b,c.0)\sum_n\frac{1}{n!}\left[\frac{2(a\xi+b\eta+c\rho)-s^2}{8\nu t}\right]^n$$

$$\times da\,db\,dc \qquad (2.35)$$

We shall consider only the first few terms of this series. For the evaluation of the contributions from different terms in the series expansion, we shall use two simple lemmas :

[Batchelor & Townsend 1948, A 194]

$$\text{(i)}\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} L(a,b,c,o)\,db\,dc=0 \qquad (2.36)$$

since the continuity requires the total flow across an infinite plane to be zero.

$$\text{(ii)}\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a^i\,b^j\,c^k\,L(a,b,c,o)\,da\,db\,dc=0 \qquad (2.37)$$

where $i+j+k$ is an odd integer. Justification to imposition of such a condition is due to the condition of homogeneity i e., $L(a,b,c,0)=L(-a,-b,-c,0)$ etc.

If we put $n=0$, it is found from (2.36) that there is no contribution to the integral (2.35). If we put $n=1$, the next significant term in the series is of order $(\nu t)^{-1/2}$, but it is evident from (2.37) that there is no contribution of this term to the integral.

We obtain the first non-zero contribution to the integral from the terms of order $(\nu t)^{-1}$ in the series. Hence, to the first approximation,

$$L(\xi,\eta,\rho,t)$$

$$=\frac{\pi\,e^{-r^2/8\nu t}}{(8\pi\nu t)^{5/2}}\int\int_{-\infty}^{\infty}\int L(a,b,c,o)\left[\frac{(a\xi+b\eta+c\rho)^2}{4\nu t}-s^2\right]da.\,db.\,dc \qquad (2.38)$$

If $r=0$, we obtain

$$1=\overline{\theta(0)\,x(0)}=\frac{-\pi\int\int_{-\infty}^{\infty}\int s^2\,L(a,b,c,0)\,da\,db\,dc}{(8\pi\nu t)^{5/2}} \qquad (2.39)$$

Now for $t=0$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^2\,L(a,b,c,0)\,da\,db\,dc$$

$$=\overline{\theta(0)\,\chi(0)}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} s^2\left[f(s,0)+\frac{s}{2}\,\frac{\partial f(s,0)}{\partial s}\sin^2\theta\right]\times s^2\sin\theta\,ds\,.\,d\theta\,d\phi$$

$$=4\pi\int_0^{\infty} s^4\left[f(s,0)+\frac{s}{3}\,\frac{\partial f(s,0)}{\partial s}\right]ds.$$

$$=-\frac{8\pi}{3}\int_0^{\infty} s^4 f(s,0)\,ds,$$

(Batchelor & Townsend 1948, p. 535)

so that (2.39) gives

$$\int_0^\infty s^4 f(s,0)\, ds = 48 \sqrt{(2\pi)}\, \nu t^{5/2} \tag{2.40}$$

(b) Let

$$Q(\xi, \eta, \rho, t) = \overline{\chi(x, y, z, t)\, \chi(x+\xi, y+\eta, z+\rho, t)}$$

$$= \underset{\substack{\alpha\to\infty \\ \beta\to\infty \\ \nu\to\infty}}{Lt} \frac{1}{8\alpha\beta\nu} \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\nu}^{\nu} \chi'(x, y, z, t)\, \chi(x+\xi, y+\eta, z+\rho, t)\, dx\, dy\, dz$$

Substituting (2.17) and (2.18) in (2.41) and proceeding in the same manner as in (2.32), we finally obtain

$$Q(\xi, \eta, \rho, t)$$

$$= \frac{\pi e^{-2c(t)}\, e^{-r^2/8\nu t}}{(8\pi\nu t)^{5/2}} \int\int_{-\infty}^{\infty}\int Q(a, b, c, 0) \left[\frac{(a\xi+b\eta+c\rho)^2}{4\nu t} - s^2\right] da\, db\, dc \tag{2.42}$$

For $r = 0$,

$$\overline{\chi^2(t)} = \frac{-\pi e^{-2c(t)} \int\int\int_{-\infty}^{\infty} s^2\, Q(a, b, c, 0)\, da\, db\, dc}{(8\pi\nu t)^{5/2}} \tag{2.43}$$

Here for $t = 0$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^2\, Q(a, b, c, 0)\, da\, db\, dc$$

$$= \overline{\chi^2(0)} \int_0^\infty\int_0^\pi\int_0^{2\pi} s^2 \left[f(s.0) + \frac{s}{2}\, \frac{\partial f(s, 0)}{\partial s}\, \sin^2\theta\right] \times s^2 \sin\theta\, ds\, d\theta\, d\phi$$

$$= -\frac{8\pi}{3}\, \overline{\chi^2(0)} \int_0^\infty s^4 f(s, 0)\, ds$$

$$= -128 \sqrt{(2)}\, \pi^{3/2} (\nu t)^{5/2}\, \overline{\chi^2(0)}$$

from (2.40). Hence

$$\overline{\chi^2(t)} = \overline{\chi^2(0)}\,.\, e^{-2c(t)} \tag{2.44}$$

Similarly,

$$\overline{\theta^2(t)} = \overline{\theta^2(0)}\,.\, e^{2c(t)}$$

(c) Let

$$R(\xi, \eta, \rho, t) = \overline{\theta_x(x, y, z, t) \cdot \theta_x'(x+\xi, y+\eta, z+\rho, t)}$$

$$= \underset{\substack{\alpha\to\infty \\ \beta\to\infty \\ \nu\to\infty}}{Lt} \frac{1}{8\alpha\beta\nu} \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\nu}^{\nu} \theta_x(x, y, z, t)\, \theta_x'(x+\xi, y+\eta, z+\rho, t)\, dx\, dy\, dz \tag{2.46}$$

Proceeding in the same manner, we obtain (2.46) as

$$R(\xi, \eta, \rho, t) = \frac{\pi e^{2c(t)}\ e^{-r^2/8\nu t}}{4\nu t(8\pi\nu t)^{5/2}} \iiint_{-\infty}^{\infty} R(a, b, c, 0) \left[1 - \frac{(\xi - a)^2}{4\nu t}\right] \times \left[\frac{(a\xi + b\eta + c\rho)^2}{4\nu t} - s^2\right] da\, db\, dc \tag{2.47}$$

[Dutta Gupta, 1975, Chapter VII].

As stated before, the first non-zero contribution to the integral comes from the terms of order $(\nu t)^{-1}$ in the series. By virtue of (2.36) and (2.37), there is no contribution to the integral for terms of order other than $(\nu t)^{-1}$. Hence, to the first approximation,

$$R(\xi, \eta, \rho, t) = \frac{\pi e^{2c(t)}\ e^{-r^2/8\nu t}}{4\nu t\ (8\pi\nu t)^{5/2}} \left(1 - \frac{\xi^2}{4\nu t}\right) \times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R(a, b, c, 0) \left[\frac{(a\xi + b\eta + c\rho)^2}{4\nu t} - s^2\right] da\, db\, dc \tag{2.48}$$

If $r = 0$, we obtain,

$$\overline{\theta_x^{\,2}(t)} = \frac{-\pi e^{2c(t)}}{4\nu t\ (8\pi\nu t)^{5/2}} \iiint_{-\infty}^{\infty} s^2\, R(a, b, c, 0)\, da\, db\, dc \tag{2.49}$$

Now for $t = 0$,

$$\iiint_{-\infty}^{\infty} s^2\, R(a, b, c, 0)\, da\, db\, dc = -128\sqrt{(2)}\ \pi^{3/2}\ (\nu t)^{5/2}\ \overline{\theta^2(0)}.$$

Hence,

$$\overline{\theta_x^{\,2}(t)} = \frac{e^{2c(t)}}{4\nu t} \overline{\theta^2(0)} \tag{2.50}$$

(d) Let

$$S(\xi, \eta, \rho, t) = \overline{\chi(x, y, z, t)\ \theta_x'(x+\xi, y+\eta, z+\rho, t)}$$

$$= \underset{\substack{\alpha \to \infty \\ \beta \to \infty \\ \nu \to \infty}}{Lt}\ \frac{1}{8\alpha\beta\nu} \int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\nu}^{\nu} \chi(x, y, z, t)\ \theta_x'(x+\xi, y+\eta, z+\rho, t)\, dx\, dy\, dz \tag{2.51}$$

The integration connected with the x-direction is

$$\underset{\alpha \to \infty}{Lt}\ \frac{1}{2\alpha} \int_{-\alpha}^{\alpha} \chi(x, t)\ \theta_x'(x+\xi, t)\, dx$$

$$= \underset{\alpha \to \infty}{Lt}\ \frac{1}{2\alpha} \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \frac{\chi(X, 0)\ \theta(X', 0)}{4\pi\nu t}\ e^{-[(X-x)^2 + (X'-x-\xi)^2]/4\nu t} \times \left(\frac{X'-x-\xi}{2\nu t}\right) dX\, dX'\, dx$$

$$= \frac{1}{8\pi(\nu t)^2} \cdot \underset{\alpha\to\infty}{\alpha t}\ \frac{1}{2\alpha} \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \chi(X,0)\,\theta(X',0)$$

$$\times\ e^{-(2x-X-X'+\xi)^2+(X-X'+\xi)^2/8\nu t}\left[-\frac{(2x-X-X'+\xi)+(X-X'+\xi)}{2}\right]$$

$$dX\,dX'\,dx$$

$$= -\frac{1}{16\pi(\nu t)^2} \cdot \underset{\alpha\to\infty}{Lt}\ \frac{1}{2\alpha} \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \chi(X,0)\,\theta(X',0)$$

$$\times\ e^{-(X-X'+\xi)^2/8\nu t}\ .\ e^{-(2x-X-X'+\xi)^2/8\nu t}\,(2x-X-X'+\xi)\,dX\,dX'\,dx$$

$$-\frac{1}{16\pi(\nu t)^2} \cdot \underset{\alpha\to\infty}{Lt}\ \frac{1}{2\alpha} \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \chi(X,0)\,\theta(X'\ 0)$$

$$\times\ e^{-(X-X'+\xi)^2/8\nu t}\ .\ e^{-(2x-X-X'+\xi)^2/8\nu t}\,(X-X'+\xi)\,dX\,dX'\,dx$$

$$= -\frac{\sqrt{(2\pi\nu t)}}{16\pi(\nu t)^2} \cdot \underset{\alpha\to\infty}{Lt}\ \frac{1}{2\alpha} \int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha} \chi(X,0)\,\theta(X',0)\,e^{-(X-X'+\xi)^2/8\nu t}$$

$$\times\ (X-X'+\xi)\,dX\,dX',$$

as the first term is zero.

$$= -\frac{1}{8\sqrt{(2\pi)}(\nu t)^{3/2}} \cdot \underset{\alpha\to\infty}{Lt} \int_{-\alpha}^{\alpha}\int_{-\alpha-X}^{\alpha-X} \chi(X,0)\,\theta(X+a,0)\,e^{-(\xi-a)^2/8\nu t}\,(\xi-a)\,dX\,da$$

where $X'=X+a$.

$$= -\frac{1}{8\sqrt{(2\pi)}(\nu t)^{3/2}} \int_{-\infty-X}^{\infty-X} \overline{\chi(X,0)\,\theta(X+a,0)}\,e^{-(\xi-a)^2/8\nu t}\,da \tag{2.52}$$

The other two sets of integration give the following results;

$$\underset{\beta\to\infty}{Lt}\ \frac{1}{2\beta}\int_{-\beta}^{\beta} x(y,t)\,\theta_{x'}(y+\eta,t)\,dy$$

$$= \frac{1}{2\sqrt{(2\pi\nu t)}}\int_{-\infty-Y}^{\infty-Y} \overline{\chi(Y,0)\,\theta(Y+b,0)}\,e^{-(\eta-b)^2/8\nu t}\,db \tag{2.53}$$

and,

$$\underset{\nu\to\infty}{Lt}\ \frac{1}{2\nu}\int_{-\nu}^{\nu} \chi(z,t)\,\theta_{x'}(z+\rho,t)\,dz$$

$$= \frac{1}{2\sqrt{(2\pi\nu t)}}\int_{-\infty-z}^{\infty-z} \overline{\chi(Z,0)\,\theta(Z+c,0)}\,e^{-(\rho-c)^2/8\nu t}\,dc \tag{2.54}$$

Combining (2.52), (2.53) and (2.54), we obtain,

$$S(\xi,\eta,\rho,t)=-\frac{1}{64\,\sqrt{(2)}\,\pi^{3/2}\,(\nu t)^{5/2}}\iiint_{-\infty}^{\infty} S(a,b,c,0)\,(\xi-a)\,e^{-[(\xi-a)^2+(\eta-b)^2+(\rho-c)^2]/8\nu t}\,da\,db\,dc=0. \tag{2.55}$$

If $r=0$, we obtain

$$\overline{\theta_x(t)\,\chi(t)}=0 \tag{2.56}$$

(e) Let $T(\xi,\eta,\rho,t)=\overline{\chi(x+\xi,y+\eta,z+\rho,t)\,\theta_x(x,y,z,t)}$

$$=\underset{\substack{\alpha\to\infty\\ \beta\to\infty\\ \alpha\to\infty}}{Lt}\frac{1}{8\alpha\beta\nu}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}\int_{-\nu}^{\nu}\chi(x+\xi,y+\eta,z+\rho,t)\,\theta_x(x,y,z,t)\,dx\,dy\,dr \tag{2.57}$$

The integration connected with the x-direction is

$$\underset{\alpha\to\infty}{Lt}\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\chi(x+\xi,t)\,\theta_x(x,t)\,dx$$

$$=\underset{\alpha\to\infty}{Lt}\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\frac{\chi(X',0)\,\theta(X,0)}{4\pi\nu t}\,e^{-[(X-x)^2+(X'-x-\xi)^2]/4\nu t}\times\left(\frac{X-x}{2\nu t}\right)dX\,dX'\,dx$$

$$=\frac{1}{8\pi(\nu t)^2}\cdot\underset{\alpha\to\infty}{Lt}\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\chi(X',0)\,\theta(X,0)\,e^{-[(2x-X-X'+\xi)^2+(X-X'+\xi)^2]/8\nu t}\times\left[\frac{(X-X'+\xi)-(2x-X-X'+\xi)}{2}\right]dX\,dX'\,dx$$

$$=\frac{1}{16\pi(\nu t)^2}\cdot\underset{\alpha\to\infty}{Lt}\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}\chi(X',0)\,\theta(X,0)\,e^{-(2x-X-X'+\xi)/8\nu t}\,.\,e^{-(X-X'+\xi)^2/8\nu t}\times(X-X'+\xi)\,dX\,dX'\,dx.$$

(the integral for other term is zero.)

$$=\frac{\sqrt{(2\pi\nu t)}}{16\pi(\nu t)^2}\cdot\underset{\alpha\to\infty}{Lt}\frac{1}{2\alpha}\int_{-\alpha}^{\alpha}\int_{-\alpha}^{\alpha}x(X',0)\,\theta(X,0)\,e^{-(X-X'+\xi)/8\nu t}.\times(X-X'+\xi)\,dX\,dX'$$

$$=\frac{1}{8\sqrt{(2\pi)}(\nu t)^{3/2}}\int_{-\infty-X}^{\infty-X}\overline{\chi(X+a,0)\,\theta(X,0)}\,e^{-(\xi-a)^2/8\nu t}\,(\xi-a)\,da \tag{2.58}$$

where $X'=X+a$.

The other two sets of integrations are obtained as (2.53) and (2.54) respectively. Finally, we obtain

$$T(\xi, \eta, \rho, t)=\frac{1}{64\sqrt{(2)}\pi^{3/2}(\nu t)^{5/2}}\int\int_{-\infty}^{\infty}\int T(a, b, c, 0)(\xi-a)$$

$$\times e^{-[(\xi-a)^2+(\eta-b)^2+(\rho-c)^2]/8\nu t}\, da\, db\, dc \tag{2.59}$$

$$=0$$

If $r=0$, we obtain

$$\overline{\chi(t)\,\theta_x(t)}=0 \tag{2 60}$$

2.6 Determination of Decay law : Let $\xi=\eta=\rho=0$ in (2.27). Then

$$\overline{u^2(t)}=4\nu^2\,\overline{[\theta_x^2(t)}\,.\,\overline{x^2(t)}+\overline{x(t)\theta_x(t)}\,.\,\overline{x(t)\,\theta_x(t)]} \tag{2.61}$$

Substituting (2.44), (2.50), (2.56) in (2.61), we obtain

$$\overline{u^2(t)}=4\nu^2\left[\frac{1}{4\nu t}\cdot e^{2c(t)}.\,\overline{\theta^2(0)}\,.\,e^{-2c(t)}.\,\overline{x^2(0)}\right]$$

$$=\frac{\nu}{t}\,\overline{\theta^2(0)}\,.\,\overline{x^2(0)} \tag{2.61A}$$

i.e.,

$$\overline{u^2(t)}\propto\frac{1}{t} \tag{2.62}$$

Similarly, we can show that

$$\overline{v^2(t)}\propto\frac{1}{t}$$

and

$$\overline{w^2(t)}\propto\frac{1}{t}\,.$$

Thus by adding all the three, we get

$$\frac{1}{2}\left[\overline{u^2(t)}+\overline{v^2(t)}+\overline{w^2(t)}\right]\propto\frac{1}{t} \tag{2.63}$$

Let us first proceed with the result :

$$\frac{1}{2}\overline{u^2}(t)\propto\frac{1}{t}$$

$$\therefore\frac{1}{1/2\overline{u^2}}\cdot\frac{\partial}{\partial t}\left(\frac{1}{2}\overline{u^2}\right)=-\frac{1}{t}$$

i.e., $-\dfrac{10\nu}{\lambda^2}=-\dfrac{1}{t}$, i.e. $\lambda^2=10\nu t$

It is worthwhile to add that similar expressions may also be obtained for $\frac{1}{2}\overline{v^2}$ and $\frac{1}{2}\overline{w^2}$. Thus equation (2.63), gives

$$\lambda^2=10\nu t \tag{2.64}$$

2.7. Conclusion: The decay law (2.64) is obtained for the period when both the inertia and viscosity terms contribute. The Millionishtchikov's quasinormality hypothesis (1941a, b) has been assumed in this case. It is found that the neglect of pressure term present in the Navier-Stokes equation does not modify the decay law (2.64). This law can also be derived from the self-preservation rule as applied to Karman-Howarth equation (1938). Heisenberg (1948a, b) obtains the same power law $\bar{u}^2 \propto t^{-1}$ for the decay of turbulence energy, when he puts forth a solution of the form $F(K, t) \sim 1/\sqrt{(t)}\, f(K\sqrt{(t)}$ for the period when both the inertia and viscosity terms are significant.

Acknowledgement: I convey my hearty thanks to Dr. K. M. Ghosh who has helped me much in the preparation of this parer.

References

Batchelor. G. K. (1953): The Theory of Homogeneous Turbulence, Cambridge University Press.

Batchelor & Townsend (1948): *Proc. Roy. Soc.* A, **194**, 527.

Cole, J. D. (1951): On a quasi-linear parabolic equation occuring in aerodynamics *Quart. App. Math*, **9**, 225.

Dutta Gupta, Santa (1975): Study of Decay Phenomena and Local and Non-local effects in Turbulence (Ph. D. dissertation).

Ghosh, K. M. (1972): Some Consequences of Millionshtechikov's hypothesis in the early period decay process of Turbulence. *Indian Journal of Pure & Applied Math.*, 3, 1.

Millionshtchikov, M. D. (1941a): On the theory of homogeneous isotropic turbulence, *C. R. Acad. Sci., S.S.S,R.* 32, 611,

— — (1941b): On the theory of homogeneous isotropic turbulence. Izv. Akad. Nauk, S.S.S.R. Series, Geogr. Geophys., 5, 433.

Sneddon, I. N. (1951): Fourier Transform, McGraw Hill Book. Co. Inc.

LECTURER IN MATHEMATICS,
K. G. ENGINEERING INSTITUTE,
P. O. BISHNUPUR, DIST. BANKURA.

Some portion of cost of publication of this number has been borne by the Department of Science and Technology, Government of India.

Bull. Cal. Math. Soc., 75, 218–222 (1983)

CONTINUITY AND FIXED IMAGE CHARACTERISATION IN GRAPH TOPOLOGICAL SPACES

TARIT KUMAR MAJUMDAR

(*Received 7 June 1979*)

1. Introduction : According to Claude Berge (Berge), a *graph* is an ordered pair (X, f) where X is a non-empty set and f is a function from X into X; f may be single-valued or multivalued.

In my previous paper (Majumdr), I have introduced the definition of Graph Topological Spaces. In that paper, the continuity of the function f, in respect to Graph Topology was considered for the case when f is single-valued. In this paper the generalised case, i.e. when f is multivalued, is also considered. Also, few results are drawn to expose the fixed image characterisation of graph. A closed basis of the Graph Topology has also been found out.

2. Preliminaries : According to my paper (Majumdar), the definitions of Graph Topological Spaces and the bases of Graph Topology are as follows :—

Definition 1. *A graph Topological Space* $(X, f; \tau)$ *is a graph* (X, f) *endowed with a topology* τ *given by*

$$\tau = \{A / A \subseteq X \text{ and } m(A) = A\}$$

where

$$m(A) = A \cup f(A) \text{ for each } A \subseteq X.$$

Result 2. The collection $\mathcal{B} = \{B_x\}_{x \in X}$

where

$$B_x = \{x \cup f(x) \cup \ldots \cup f^n(x) \cup \quad \}$$

forms an irreducible open base for the Graph Topology τ and for each $x \in X$ B_x solely constitutes a local base at x for the Graph Topology.

Here also, we require to adopt the following definitions of continuity of a multivalued function, given by Claude Berge (Berge).

Definition 3. *Let* $f : X \rightarrow Y$ *be a multi-valued function (may be single-valued) where* X *and* Y *are topological spaces. Then,*

(i) f is said to be *lower-semi-continuous* (l.s.c) at a point $x_0 \in X$, if for each open set G *meeting* $f(x_0)$, there exists a *ngd*. $\mathcal{U}(x_0)$ of x_0 such that for each $x \in \mathcal{U}(x_0)$, $f(x) \cap G \neq \phi$

(ii) f is said to be *upper-semi-continuous* (u.s.c.) at a point $x_0 \in X$, if for every open set G *containing* $f(x_0)$, there exists a *ngd.* $U(x_0)$, of x_0 such that for each x belonging to $U_{x_0}, f(x) \subseteq G$.

(iii) f is said to be *continuous at* x_0 if f is both u.s.c and l.s.c. at x_0.

(iv) f is said to be *continuous in* X, if f is continuous at every point of X and *moreover* $f(x)$ is compact for each point $x \in X$.

Note 4. The definition of continuity of a multi-valued function $f: X \to Y$ was given separately by Claude Berge (Berge), K. Kuratowski (1932) and Bouligand (1932).

Berge defined the local continuity first, and then generalised it to a global concept, by taking the images of f compact. But Kuratowski took the global concept first and then restricted it to have the local continuity. Certainly therefore, Berge's definition differs from that of Kuratowski whenever one deals with non-compact spaces and also whenever one considers the case that the image of a point $x \in X$, under f can be empty. The latter case is of course of no importance to our view-point.

3. Continuity in Graph Topological Spaces:

THEOREM 1. *If* $(X, f; \tau)$ *be a Graph Topological Space*; $f: X \to X$ *is continuous at every point of* X.

Proof: Let $x_0 \in X$ be arbitrary and G be an open set in $(X, f: \tau)$, such that

$$G \cap f(x_0) \neq \phi$$

$$\Rightarrow f[G \cap f(x_0)] \neq \phi$$

$$\Rightarrow f^2(x_0) \cap f(G) \neq \phi \; [\because f(G) \subseteq G \text{ as } G \text{ is open}]$$

Similarly, $f^3(x_0) \cap G \neq \phi$ and so on.

Thus, for any $m \geqslant 1$, $f^m(x_0) \cap G \neq \varphi$.

Now, $x \in B_{x_0}$

$$\Rightarrow x \in f^{m-1}(x_0) \text{ for some } m \geqslant 1$$

$$\Rightarrow \quad f(x) \subseteq f^m(x_0)$$

$$\Rightarrow \quad f(x) \cap G \neq \phi$$

$$\Rightarrow \quad f \text{ is l.s.c. at } x_0.$$

Next, let H be an open set in $(X, f; \tau)$ such that $f(x_0) \subseteq H$.

$$\Rightarrow f^m(x_0) \subseteq H \text{ for all integer } m \geqslant 1.$$

$$\Rightarrow \text{for any } x \in B_{x_0}; f(x) \subseteq H$$

$$\Rightarrow f \text{ is u.s.c. at } x_0.$$

This completes the proof.

Note 2. Certainly, according to Claude Berge (Berge), f is continuous in X, if for each $x \in X, f(x)$ is compact. This is possible, if $f(x)$ is finite for each x or more generally, $f(x) \subseteq \bigcup_{i=1}^{n} B_{x_i}$ for some $x_i \in f(x)$ $(i=1, 2, 3, \ldots n)$ (vide Result-21, Sec.-2, Majumdar).

Note 3. According to Berge (Berge), (vide Th. 3. p-110). We get the following result.

Result 4. *If Γ be u.s.c. from $X \to Y$, then $\supset \Gamma K$ is compact if K is compact.*

The proof of the above result obviously depends on the fact that $f(x)$ is compact for each $x \in X$. In the Graph Topological Space, the condition is not necessary. However, we may have the analogous result. Before establishing that, we require the following definitions and lemmas:—

Definition 5. (Due to Berge—)

The lower inverse $\overline{f}$ of a mapping $f: X \to Y$, is defined by

$$\overline{f}(y) = \{x / x \in X, y \in f(x)\} \,;\, y \in Y.$$

Thus for a non-empty subset

$$B \subseteq Y \;\; \overline{f}(B) = \{x / x \in X, f(x) \cap B \neq \phi\}$$

LEMMA 6. *If $f: X \to Y$. Then for each $x \in X, x \in \overline{f} f(x)$.*

Proof. Obvious, as $x \notin \overline{f} f(x)$ follows a contradiction.

LEMMA 7. *In a Graph Topological Space $(X, f: \tau): \overline{f}(B_x)$ is open for each $x \in X$.*

Proof. For any $x_0 \in \overline{f}(B_x), f(x_0) \cap B_x \neq \phi$ by definition 5

$\Rightarrow f^n(x_0) \cap B_x \neq \phi$ for all integer $n \geqslant 1$.

$\Rightarrow$ For all $y \in B_{x_0}, f(y) \cap B_x \neq \phi$ and so $y \in \overline{f}(B_x) \Rightarrow B_{x_0} \subseteq \overline{f}(B_x)$

And hence $\overline{f}(B_x)$ is a *ngd.* of each point $x_0 \in \overline{f}(B_x)$

This completes the proof.

LEMMA 8. *In the Graph Topological Space $(X, f; \tau); \overline{f}(B_x) \subseteq B_x$ for each $x \in X$.*

Proof: If $y \notin B_x$, we have $y \notin f^m(x)$ for all integer $m \geqslant 0$

$\Rightarrow f(y) \cap B_x = \phi$

$\Rightarrow y \notin \overline{f}(B_x)$.

This completes the proof.

THEOREM 9. *In a Graph Topological Space, $f(K)$ is compact, if K is compact in X.*

Proof: Let us consider the covering $\{B_x / x \in f(K)\}$ by the basic open sets in $(X, f; \tau$.

$$\Rightarrow \bigcup_{x \in f(K)} B_x \supseteq f(K).$$

$$\Rightarrow \overline{f}\Big[\bigcup_{x \in f(K)} B_x\Big] \supseteq K \text{ (by lemma-6.)}$$

$$\Rightarrow \bigcup_{x \in f(K)} \overline{f}(B_x) \supseteq K.$$

As K is compact, it follows from lemma-7

that $\bigcup_{i=1}^{n} \overline{f}(B_{x_i}) \supseteq K$ where $x_i \in f(K)$ for $i=1, 2, \cdots n$.

$$\Rightarrow \bigcup_{i=1}^{n} f\overline{f}(B_{x_i}) \supseteq f(K)$$

$$\Rightarrow \bigcup_{i=1}^{n} \overline{f}(B_{x_i}) \supseteq f(K) \quad [\because f(G) \subseteq G \text{ for any } G \in \tau]$$

$$\Rightarrow \bigcup_{i=1}^{n} B_{x_i} \supseteq f(K) \quad \text{[By Lemma-8]}$$

This completes the proof.

4. Fixed Image Characteristics of Graph : The following result is obvious.

Result 1. *If for each $x \in X$, there exists a +ve integer n such that $x \in f^n(x)$, then $f(G)=G$ for each open G and conversely.*

Definition 2. *Let $(X, f; \tau)$ be a Graph Topological Space. A subset $K \subseteq X$ is said to be a k-set iff*

$$K=\{z / \overline{f}(z) \cap K \neq \phi\}$$

Note 3. It follows from Definition-2, that if $K \subseteq X$ is a k-set then $z \in K$ iff there exists an $x \in K$ for which $z \in f(x)$.

Result 4. *In a Graph Topological Space $(X, f; \tau)$, a set $K \subseteq X$ is a k-set iff for each $x \in K$, there exists a sequence of points $x_1, x_2 \ldots x_n$. in K, such that $x \in f^n(x_n)$ for each positive integer n.*

The proof of result-4 is made on the induction described in Note-3.

Result 5. *If in a Graph Topological Space $(X, f; \tau)$ f is injective, then every k-set is closed.*

Proof. Let K be a k-set and p be a limit point of K. As B_p solely constitutes a local base at p, it follows that there exists an $x \in K\,(x \neq p)$ such that $x \in f^n(p)$ for some positive integer $n > 0$. Again by result-4, there exists an $x_n \in K$ such that $x \in f^n(x_n)$

$$\Rightarrow f^n(p) \cap f^n(x_n) \neq \phi$$

$$\Rightarrow x_n = p \in K$$

This completes the proof.

Note 6. In proving result-5, we have considered the following definition due to Berge.

Definition. *A function $f: X \rightarrow Y$ is said to be injective iff for any*

$$x \neq y, f(x) \cap f(y) = \phi$$

Result 7. *All k-sets of a Graph Topological Space are open.*

Proof. For any k-set K in the Graph Topological Space $f(K) \subseteq K$ and hence K is open.

Result 8. *The k-sets in a Graph Topological space $(X, f; \tau)$ are kept fixed under f and hence image of a k-set under f is again a k-set.*

Proof. If $K \subseteq X$ be a k-set we notice $K \subseteq f(K)$ by result 4 and hence by Result-7, $K = f(K)$.

This completes the proof.

5. Closed Basis of Graph Topology.

Definition 1. *Let $(X. f; \tau)$ be a Graph Topological Space. A subset C of X is said to be a c-set iff*

$$C = \{z / B_z \supseteq B_x \text{ for some } x \in C\}$$

Result 2. *Every c-set is closed*

Proof. Let $C \subseteq X$ be a c-set and p be a limit point of C. Then there exists an $x \in C$, such that $x \in B_p \Rightarrow B_p \supseteq B_p \supseteq B_x \Rightarrow p \in C$.

Result 3. *Every closed set in a Graph Topological Space $(X, f; \tau)$ is the union of some c-sets.*

Proof. Let F be a closed set in $(X, f; \tau)$. Define for each

$$x \in F,\ C_x = \{y / B_y \supseteq B_x\}.$$

Then C_x is a c-set for each x and

$$x \in c_x \subseteq F \Rightarrow F = \bigcup_{x \in F} C_x$$

This proves the result.

Note 4. It follows from the previous result, that the collection

$$\mathcal{C} = \{C_x / x \in X\}$$

forms a closed basis of the Graph Topological Space $(X, f; \tau)$.

Acknowledgement. My heart-felt thanks to Dr. S. Ganguly, Calcutta University for his valuable guidance in the preparation of this paper.

References

Bouligand (1932): *Ens. Math.* 14.

Berge Claude: Topological Spaces Translated by E. M. Patterson,

— — : Theory of Graphs and its application. Translated by Alison Doig.

Kuratowski K. (1932): *Fund. Math.* 1, p-148.

Majumder Tarit K.: Graph Topologies on a set. Communicated to the Bulletin of Calcutta Mathematical Society, India.

JR. RESEARCH FELLOW (U.G.C.)
UNIVERSITY OF CALCUTTA,
DEPT. OF PURE MATHEMATICS,
35, BALLYGUNGE CIRCULAR ROAD,
CALCUTTA-700019, INDIA.

Bull. Cal. Math. Soc., 75, 223—226 (1983)

ON THE ABSOLUTE CESÀRO SUMMABILITY OF A SERIES ASSOCIATED WITH THE DERIVED FOURIER SERIES

M. D. CHATURVEDI

(*Received 9 June 1979*)

Abstract : The paper is devoted to study the absolute Cesàro summability of a series associated with the derived Fourier series of a Lebesgue integrable function.

1. Definitions and Notations : *Let* $\sum u_n$ *be a given infinite series. Let the* n^{th} *Cesàro means of order* α *of the sequence*

$$\{S_n\}, \left(S_n = \sum_{v=1}^{n} u_v\right) \text{ and } \{nu_n\}$$

be denoted by S_n^α *and* τ_n^α *respectively.*

The series $\sum u_n$ is said to be absolutely summable (C, α) or summable $|C, \alpha|$, $\alpha > -1$, if the series

$$\sum \frac{|\tau_n^\alpha|}{n}$$

is convergent.

Let $f(t) \in L(-\pi, \pi)$ be a 2π-periodic function, and let its Fourier series be given by

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t). \tag{1.1}$$

Then the series obtained after differentiating the above Fourier series is

$$\sum_{n=1}^{\infty} n(b_n \cos nt - a_n \sin nt) = \sum_{n=1}^{\infty} nB_n(t). \tag{1.2}$$

The series

$$\sum \frac{S_n(x)}{n} \tag{1.3}$$

will be called the associated derived Fourier series, where $S_n(x)$ denotes the n^{th} partial sum of the series (1.2).

We shall use the following notations:

$$D_n(t) = \frac{\sin(n+1/2)t}{2\sin t/2},$$

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\},$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u)\, du, \qquad \alpha > 0,$$

$$\Phi_0(t) = \phi(t),$$

$$\phi_\alpha(t) = \Gamma(1+\alpha)\, t^{-\alpha}\, \Phi_\alpha(t) \qquad \alpha \geqslant 0.$$

We employ $\psi_\alpha(t)$ and $\psi_\alpha(t)$ with similar meanings.

2. **Introduction** : We prove the following result pertaining to $|C, \beta|$ summability of the series (1.3). The result obtained is to some extent an analogue of a result of Mazhar (1967) for associated Fourier series. More precisely, we prove the following theorem :

THEOREM : *Let*

$$\psi_\alpha(t) = 0 \text{ and } \int_0^\pi t^{-\alpha-1} |d\,\psi_\alpha(t)| < \infty.$$

Then the series (1.3) *is summable* $|C, \beta|$, $\beta > \alpha \geqslant 0$.

3. *Proof of the theorem.* We have

$$S_n(x) = -\frac{2}{\pi} \int_0^\pi \psi(t) \frac{d}{dt} D_n(t)\, dt.$$

Let τ_n^β and $K_n^\beta(t)$ denote the n^{th} Cesàro means of order β of $S_n(x)$ and $D_n(t)$ respectively.

It is known [Pati 1954 & Zygmund 1935] that

$$\left(\frac{d}{dt}\right)^\rho K_n^\beta(t) = \begin{cases} O(n^{\rho+1}),\ 0 < t \leqslant 1/n, \\ O(t^{-1-\beta} n^{\rho-\beta}) + O(n^{-1} t^{-\rho-2}),\ 1/n < t \leqslant \pi, \end{cases} \tag{3.1}$$

where ρ is a zero or a positive integer and $\beta \geqslant 0$.

We assume, as we may, without any loss of generality $0 \leqslant \alpha < \beta < h+1$, where h is the greatest integer less than α.

Now, by integrating by parts we have

$$-\frac{\pi}{2} \tau_n^\beta = \int_0^\pi \psi(t) \frac{d}{dt} K_n^\beta(t)\, dt$$

$$= \left[\sum_{\rho=1}^{h} (-1)^{\rho-1} \psi_\alpha(t) \left(\frac{d}{dt}\right)^\rho K_n^\beta(t) \right]_0^\pi$$

$$+ (-1)^{h+1} \int_0^\pi \psi_h(t) \left(\frac{d}{dt}\right)^{h+1} K_n^\beta(t)\, dt$$

$$= 0(n^{\alpha-\beta}) + (-1)^{h+1} R_n,$$

where

$$R_n = \int_0^{\pi} \psi_h (t) \left(\frac{d}{dt}\right)^{h+1} K_n^{\beta} (t)\, dt$$

$$= \frac{1}{\Gamma(h+1-\alpha)} \int_0^{\pi} \left(\frac{d}{dt}\right)^{h+1} K_n^{\beta} (t)\, dt \int_0^{t} (t-u)^{h-\alpha}\, d\psi_\alpha (u)$$

$$= \frac{1}{\Gamma(h+1-\alpha)} \int_0^{\pi} d\,\psi_\alpha (u) \int_u^{\pi} (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^{h+1} K_n^{\beta} (t)\, dt.$$

Thus to complete the proof of our theorem it is sufficient to show that

$$\sum \frac{|\tau_n^{\beta}|}{n} < \infty. \tag{3.2}$$

To prove the above it is enough to show that

$$\int_0^{\pi} | d\,\psi_\alpha (u) | \sum \frac{|J(n, u)|}{n} < \infty, \tag{3.3}$$

where

$$J(n, u) = \frac{1}{\Gamma(h+1-\alpha)} \int_u^{\pi} (t-u)^{h-\alpha} \left(\frac{d}{dt}\right)^{h+1} K_n^{\beta} (t)\, dt.$$

Again (3.2) holds provided that

$$\sum \frac{|J(n, u)|}{n} = O(u^{-\alpha-1}). \tag{3.4}$$

We now proceed to prove (3.4). Writing

$$J(n, u) = \int_u^{u+n^{-1}} + \int_{u+n^{-1}}^{\pi},$$

in view of estimates (3.1), one can prove that

$$J(n, u) = \begin{cases} O(n^{\alpha+1}) \\ O(n^{\alpha-\beta}\, u^{-\beta-1}) \end{cases}$$

Thus

$$\sum \frac{|J(n, u)|}{n} = \sum_{n \leqslant u^{-1}} + \sum_{n > u^{-1}}$$

$$= \sum_{n \leqslant n^{-1}} O(n^{\alpha}) + \sum_{n > u^{-1}} O(n^{\alpha-\beta-1}\, u^{-\beta-1})$$

$$= O(u^{-\alpha-1}).$$

This completes the proof of (3.4). Thus the theorem follows from (3.3), (3.4) and the hypothesis.

Acknowledgement : The author is thankful to Dr. R. N. Singh for his kind encouragement and would like to express his thanks to the referee for his helpful suggestions to improve the presentation of this paper.

References

Kogbetliantz, E, (1925), Sur les séries absolument sommables par la méthode des moyennes arithmétiques., *Bulletin des Sciences Mathematiques* (2) 49, 234.

Kogbetliantz, E. (1931), *Sommation des séries et integrales divergentes par les moyennes arithmétiques et typiques*, Mémorial des Sciences Mathematiques; 51.

Mazhar, S. M., (1967); *Math. Scand*. 21, 90.

Pati, T., (1954), *Trans. Amer. Math. Soc.*, 76. 351.

Zygmund, A. (1935), *Trigonometrica Series.*

DEPARTMENT OF MATHEMATICS,
AVADH UNIVERSITY,
FAIZABAD.

Bull. Cal. Math. Soc., **75**, 227—231 (1983)

GROUP-THEORETIC STUDY OF CERTAIN GENERATING FUNCTIONS OF JACOBI POLYNOMIALS

BANDANA GHOSH

(*Received 24 March 1983*)

Abstract : Making suitable interpretations to the index and the parameters of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in order to derive the elements of Lie algebra, we have considered five parameter Lie group for this polynomial which does not seem to appear before. By means of this group theoretic method some new generating function^S for Jacobi polynomials are obtained from which several special generating functions can be easily derived.

1. Introduction :

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, defined by

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n\,!}\;{}_2F_1\left[\begin{matrix}-n,\ 1+\alpha+\beta+n\ ;\ \frac{1-x}{2}\\ 1+\alpha\ ;\end{matrix}\right] \tag{1.1}$$

is the solution of the ordinary differential equation :

$$(1-x^2)\frac{d^2v}{dx^2}+[\beta-\alpha-(2+\alpha+\beta)x]\frac{dv}{dx}+n(1+\alpha+\beta+n)\,v=0. \tag{1.2}$$

Various generating functions for Jacobi polynomials were derived by E. Feldheim (1943), W. A. Al-Salam (1964), M. K. Das (1972) and others.

The object of the present paper is to derive some generating functions of Jacobi polynomials by interpreting n, α, β simultaneously with the help of Weisner's group-theoretic method (Mcbride 1971).

Here the following generating functions are derived for $P_n^{(\alpha,\beta)}(x)$ by finding a set of infinitesimal operators A_i ($i=1, 2, 3, 4, 5$) constituting a Lie algebra :

$$P_n^{(\alpha,\beta)}(x+t_1)=\sum_{p=0}^{n}\frac{1}{2^p}\frac{1}{p\,!}(\alpha+\beta+n+1)_p\;P_{n-p}^{(\alpha+p,\beta+p)}(x)\,t_1^p. \tag{1.3}$$

$$\times[1-t_2(1+x)]^{\alpha}[1+t_2(1-x)]^{\beta}P_n^{(\alpha,\beta)}[x+t_2(1-x^2)]$$
$$=\sum_{k=0}^{\infty}(-2)^k/k\,!\,(n+1)_k\;P_{n+k}^{(\alpha-k,\beta-k)}(x)\,t_2^k. \tag{1.4}$$

$$[1-\omega(1+x)]^{\alpha}[1+\omega(1-x)]^{\beta}P_n^{(\alpha,\beta)}[(1-\omega x)\{x(1+1/\omega_1)-1/\omega_1\omega\}+\omega(1+1/\omega_1)] \tag{1.5}$$

$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(-1)^k}{k\,!}\frac{(-1/\omega_1)^p}{p\,!}2^{k-p}(\alpha+\beta+n+1)_p\,(n-p+1)_k\;P_{n-p+k}^{(\alpha+p-k,\beta+p-k)}(x)\,\omega^{k-p}$$

$$\times[1+(u_2/2)(1+x)]^{\alpha}[1+(u_2/2)(x-1)]^{\beta}=\sum_{k=0}^{\infty}P_k^{(\alpha-k,\beta-k)}(x)\,u_2^k. \tag{1.6}$$

2. Group theoretic method : Replacing d/dx by $\partial/\partial x$, α by $y\,\partial/\partial y$, β by $z\,\partial/\partial z$, n by $t\,\partial/\partial t$ and v by $u\ (x, y, z, t)$ we get the following partial differential equation.

$$(1-x^2)\frac{\partial^2 u}{\partial x^2} + z\,(1-x)\frac{\partial^2 u}{\partial z\,\partial x} - y\,(1+x)\frac{\partial^2 u}{\partial y\,\partial x} + ty\frac{\partial^2 u}{\partial t\,\partial y} + tz\frac{\partial^2 u}{\partial t\,\partial z} + t^2\frac{\partial^2 u}{\partial t^2} - 2\,x\frac{\partial u}{\partial x} + 2t\frac{\partial u}{\partial t} = 0. \tag{2.1}$$

Thus $u_1\,(x, y, z, t) = P_n^{(\alpha,\beta)}(x)\, y^{\alpha}\, z^{\beta}\, t^{n}$ is a solution of the differential equation (2.1) since $P_n{}^{(\alpha,\beta)}(x)$ is a solution of (1.2). Let us define the infinitesimal operators $A_i (i=1, 2, 3, 4, 5)$:

$$A_i = A_i^{(1)}\,\partial/\partial x + A_i^{(2)}\,\partial/\partial y + A_i^{(3)}\,\partial/\partial z + A_i^{(4)}\,\partial/\partial t + A_i^{(0)}$$

as follows

$$\begin{aligned}
A_1 &= y\,\partial/\partial y\\
A_2 &= z\,\partial/\partial z\\
A_3 &= t\,\partial/\partial t\\
A_4 &= yzt^{-1}\partial/\partial x\\
A_5 &= (1-x^2)\,y^{-1}\,z^{-1}\,t\,\partial/\partial x - (1+x)\,z^{-1}\,t\,\partial/\partial y + (1-x)\,y^{-1}\,t\,\partial/\partial z
\end{aligned} \tag{2.2}$$

which satisfy the following rules

$$\begin{aligned}
A_1\,[\,P_n^{(\alpha,\beta)}\,(x)\,y^{\alpha}\,z^{\beta}\,t^{n}\,] &= \alpha\,P_n^{(\alpha,\beta)}\,(x)\,y^{\alpha}\,z^{\beta}\,t^{n}\\
A_2\,[\,P_n^{(\alpha,\beta)}\,(x)\,y^{\alpha}\,z^{\beta}\,t^{n}\,] &= \beta\,P_n^{(\alpha,\beta)}\,(x)\,y^{\alpha}\,z^{\beta}\,t^{n}\\
A_3\,[\,P_n^{(\alpha,\beta)}\,(x)\,y^{\alpha}\,z^{\beta}\,t^{n}\,] &= n\,P_n^{(\alpha,\beta)}\,(x)\,y^{\alpha}\,z^{\beta}\,t^{n}\\
A_4\,[\,P_n^{(\alpha,\beta)}\,(x)\,y^{\alpha}\,z^{\beta}\,t^{n}\,] &= \tfrac{1}{2}\,(\alpha+\beta+n+1)\,P_{n-1}^{(\alpha+1,\beta+1)}(x)\,y^{\alpha+1}\,z^{\beta+1}\,t^{n-1}\\
A_5\,[\,P_n^{(\alpha,\beta)}\,(x)\,y^{\alpha}\,z^{\beta}\,t^{n}\,] &= -2(n+1)\,P_{n+1}^{(\alpha-1,\beta-1)}\,(x)\,y^{\alpha-1}\,z^{\beta-1}\,t^{n+1}.
\end{aligned}$$

Now we shall find the commutator relations by using the commutator notation with

$$[\,A, B\,]\,u = (AB-BA)\,u.$$

We have

$$\begin{aligned}
&[\,A_1, A_2\,] = 0\,;\quad [\,A_2, A_3\,] = 0\,;\quad [\,A_3, A_4\,] = -A_4\\
&[\,A_1, A_3\,] = 0\,;\quad [\,A_2, A_4\,] = A_4\,;\quad [\,A_3, A_5\,] = A_5\\
&[\,A_1, A_4\,] = A_4\,,\\
&[\,A_1, A_5\,] = -A_5\,;\quad [\,A_2, A_5\,] = -A_5\,;\quad [\,A_4, A_5\,] = -(A_1+A_2)
\end{aligned}$$

3. Lie Algebra : So we see from the above commutator relations that the set of operators $\{\,1, A_i,\ i = 1, 2, 3, 4, 5\,\}$ generates a Lie algebra.

Now the partial differential operator L, given by

$$L = (1-x^2)\frac{\partial^2}{\partial x^2}+z(1-x)\frac{\partial^2}{\partial z\partial x}-y(1+x)\frac{\partial^2}{\partial y\partial x}$$
$$+ty\frac{\partial^2}{\partial t\partial y}+tz\frac{\partial^2}{\partial t\partial z}+t^2\frac{\partial^2}{\partial t^2}-2x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}.$$

which can be expressed as

$$L = A_5 A_4+A_3(A_1+A_2+A_3+1), \tag{3.1}$$

commutes with A_i $(i = 1, 2, 3, 4, 5)$

$$\text{i. e.} \quad [A_i, L] = 0. \tag{3.2}$$

The extended form of the groups generated by A_i $(i = 1, 2, 3, 4, 5)$ are given by

(1) $e^{a_1 A_1} u(x, y, z, t) = u(x, e^{a_1} y, z, t)$

(2) $e^{a_2 A_2} u(x, y, z, t) = u(x, y, e^{a_2} z, t)$

(3) $e^{a_3 A_3} u(x, y, z, t) = u(x, y, z, e^{a_3} t)$

(4) $e^{a_4 A_4} u(x, y, z, t) = u\left[\frac{tx+a_4 yz}{t}, y, z, t\right].$

(5) $e^{a_5 A_5} u(x, y, z, t) = u\left[\frac{xyz+a_5 t(1-x^2)}{yz}, \frac{yz-a_5 t(1+x)}{z}, \frac{yz+a_5 t(t-x)}{y}, t\right]$

Therefore we easily get

$$e^{a_5 A_5} e^{a_4 A_4} e^{a_3 A_3} e^{a_2 A_2} e^{a_1 A_1} f(x, y, z, t) = f(\xi, \eta, \rho, \theta), \tag{3.3}$$

where,

$$\xi = \frac{xyzt+a_5 t^2(1-x^2)+a_4\{(yz-a_5 tx)^2-a_5^2 t^2\}}{yzt},$$

$$\eta = \frac{e^{a_1}\{yz-a_5 t(1+x)\}}{z},$$

$$\rho = e^{a_2}\left\{\frac{yz+a_5 t(1-x)}{y}\right\},$$

$$\theta = e^{a_3} t.$$

It may be of interest to remark that, by virtue of the commutator relations given above,

$$\exp(a_5 A_5+a_4 A_4+a_3 A_3+a_2 A_2+a_1 A_1)$$
$$\neq \exp(a_5 A_5)\exp(a_4 A_4)\exp(a_3 A_3)\exp(a_2 A_2)\exp(a_1 A_1) \tag{3.4}$$

The relation (3.3) is obtained by using the operator mentioned in the Right side of (3.4). Thus the order of A_i $(i = 1, \dots, 5)$ can be changed at ease without altering the effect in the left member of (3.4), while that can not be changed in the right member of (3.4). So if we change the order of the operator mentioned in the right side of (3.4) the relation (3.3) will be changed.

4. **Generating Functions:** From (2.1), $u(x, y, z, t) = P_n^{(\alpha,\beta)}(x)\, y^{\alpha}\, t^{\beta}\, t^n$ is a solution of the systems

$$\begin{cases} Lu = 0 \\ (A_1 - \alpha)\, u = 0\,; \end{cases} \qquad \begin{cases} Lu = 0 \\ (A_2 - \beta)\, u = 0\,; \end{cases} \qquad \begin{cases} Lu = 0 \\ (A_3 - n)\, u = 0\,; \end{cases}$$

$$\begin{cases} Lu = 0 \\ (A_1 + A_2 + A_3 - \alpha - \beta - n)\, u = 0. \end{cases}$$

From (3.2), we easily get

$$SL\,(P^{(\alpha,\beta)}(x)\, y^{\alpha}\, z^{\beta}\, t^n) = LS\,(P_n^{(\alpha,\beta)}(x)\, y^{\alpha}\, z^{\beta}\, t^n) = 0\,.$$

where

$$S = e^{a_5 A_5}\, e^{a_4 A_4}\, e^{a_3 A_3}\, e^{a_2 A_2}\, e^{a_1 A_1}.$$

Therefore the transformation $S\,[\,P_n^{(\alpha,\beta)}(x)\, y^{\alpha}\, z^{\beta}\, t^n\,]$ is annulled by L.

By setting $a_1 = a_2 = a_3 = 0$ in (3.3), we get

$$\begin{aligned} & e^{a_5 A_5}\, e^{a_4 A_4}\,[\,P_n^{(\alpha,\beta)}(x)\, y^{\alpha}\, z^{\beta}\, t^n\,] \\ &= \left[\frac{yz - a_5\, t(1+x)}{z}\right]^{\alpha} \left[\frac{yz + a_5\, t(1-x)}{y}\right]^{\beta} t^n \\ &\quad \times P_n^{(\alpha,\beta)}\left[\frac{xyzt + a_5\, t^2(1-x^2) + a_4\,\{(yz - a_5 tx)^2 - a_5^2\, t^2\}}{yzt}\right] \\ &= \sum_{k=0}^{\infty} \sum_{p=3}^{n+k} \frac{(a_5)^k}{k\,!}\, \frac{(a_4)^p}{p\,!}\, \{\alpha+\beta+n+1)_p\, (-1)^k\, (2)^{k-p}\, (n-p+1)_k \\ &\quad \times P_{n-p+k}^{(\alpha+p-k,\ \beta+p-k)}(x)\, y^{\alpha+p-k}\, z^{\beta+p-k}\, t^{n-p+k}. \end{aligned} \tag{4.1}$$

If we change the order of $e^{a_5 A_5}\, e^{a_4 A_4}$ we shall get the relation different from (4.1).

Now we consider the following three cases:

Gase 1:

Letting $a_4 = 1$, $a_5 = 0$ and writting $yz/t = t_1$ in (4.1) we get

$$P_n^{(\alpha,\beta)}(x + t_1) = \sum_{p=0}^{n} \frac{1}{(2)^p}\, \frac{1}{p\,!}\, (\alpha+\beta+n+1\}_p\, P_{n-p}^{(\alpha+\beta,\ \beta+p)}(x)\, t_1^{\,p} \tag{4.2}$$

which is (1.3).

Case 2 :

Putting $a_5 = 1,\ a_4 = 0$ and writing $t/yz = t_2$ in (4.1) we get

$$[\,1-t_2\,(1+x)\,]^{\alpha}\,[\,1+t_2\,(1-x)\,]^{\beta}\,P_n^{(\alpha,\beta)}\,[\,x+t_2\,(1-x^2)\,]$$
$$=\sum_{k=0}^{\infty}\frac{(-2)^k}{k\,!}\,(n+1)_k\,P_{n+k}^{(\alpha-k,\beta-k)}\,(x)\,t_2^{\,k}. \tag{4.3}$$

which is (1.4).

Case 3 :

Finally substituting $a_5 = 1, a_4 = -1/w_1$ and putting $t/yz = w$ in (4.1) we get

$$[\,1-w\,(1+x)\,]^{\alpha}\,[\,1+w\,(1-x)\,]^{\beta}\,P_n^{(\alpha,\beta)}\,[\,(1-wx)\,\{\,x(1+1/w_1)-1/w_1\ w\,\}$$
$$+\,w(1+1/w_1)\,]$$

$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(-1)^k}{k\,!}\,\frac{(-1/w_1)^p}{p\,!}\,2^{k-p}(\alpha+\beta+n+1)_p\,(n-p+n+1)_k$$
$$\times\,P_{n-p+k}^{(\alpha+p-k,\ \beta+p-k)}(x)\,w^{k-p} \tag{4.4}$$

which is (1.5).

5. Derivation of the relation (1.6). Putting $n = 0$ and $2t_2 = -u_2$ in (4.3) we get

$$\left[1+\frac{u_2}{2}\,(1+x)\right]^{\alpha}\left[1+\frac{u_2}{2}\,(x-1)\right]^{\beta}=\sum_{k=0}^{\infty}P_k^{(\alpha-k,\ \beta-k)}\,(x)\,u_2^{\,k}\,, \tag{5.1}$$

which is a result of a paper of W. A. Al-Salam (1964).

Acknowledgement : I am indebted to Dr. S. K. Chatterjea for his kind help in preparing this work.

References

Al-Salam, W. A. (1964) : Operational Representations for the Laguerre and other polynomials, *Duke Math. Jour.*, **31**, 127.

Das, M. K. (1972) : Some properties of special functions derived from the theory of continuous transformation group, *Proceedings of the American Mathematical Society*, 35, (2)565.

Feldheim, E. (1943) : Relations entreles polynomes de Jacobi, Laguerre et Hermite, *Acta Mathematica*, 75, 117.

Mcbride, E. B. (1971) : Obtaining generating functions, Springer-verlag, Berlin.

DAKSHIN BARASAT SRI SRI SARADAMANI BALIKA VIDYALAYA
P. O. & VILL.—DAKSHIN BARASAT
DT. 24-PARGANAS, WEST BENGAL, INDIA.

Bull Cal. Math. Soc, 75 232 238, (1983)

THE STABILITY OF THE DISTURBED MOTION OF A TWISTED STRIP UNDER CENTRIFUGAL AND AERODYNAMIC FORCE FIELD

B. B. SHARMA

(*Received 7 July 1979*; *Revision Received 18 March 1981*)

Abstract : In this paper the author has discussed the stability of the disturbed motion of a coupled torsional and longitudinal vibrations of a pretwisted slender beam under aerodynamic coupling in a steady flow of an incompressible fluid in a centrifugal force field. The determinental equation and the conditions of stable motion have been obtained. Galarkin's method has been used to obtain the approximate solution. It has further been shown that the critical flutter condition is exactly the critical flutter equation (13) (Sharma 1973). A numerical example has been considered to discuss the conditions mentioned above.

Nomenclature :

E = the Younge's modulus of elasticity

l = length of the beam

G = Shear modulus of rigidity

J = a constant depending upon the cross section of the beam

α = the angle through which each cross-section rotates

h = the displacement in positive x-direction

ρ = the density of the fluid

β = the twist in radians per unit length of x-axis

S = the area of the cross-section of the beam

$I_0 = \int_S r^2 dS$

$I_1 = \int_S r^4 dS$

r = the distance of longitudinal fibre from the axis of the bar

C_L = Lift coefficient (Sharma 1978)

C_N = Moment coefficient

x_0 = the distance of elastic axis after the leading edge

c = the chord length of the cross-section (air foil)

t = time

Introduction : The analysis presented in this paper considers the stability of the disturbed motion of a pretwisted slender beam that could represent a turbine blade of simple geometry. The beam is attached to a disc of radius r_0 and the disc rotates with a constant angular velocity Ω (Fig. 1). It is well known that when a thin bar is under torsion there is a slight decrease in distance between cross-sections. A normal stress is produced in each longitudinal fibre which is not parallel to the axis of the bar, and hence there is a stress component which produces an additional torque.

The Differential Equations and discussion of the problem : The governing differential equations for the coupled torsional and longitudinal vibrations of a pretwisted slender beam are

$$ES\,\frac{\partial^2 h}{\partial x^2}+EI_0\,\beta\,\frac{\partial^2 \alpha}{\partial x^2}-\rho S\,\frac{\partial^2 h}{\partial t^2}=0$$

$$(GJ+EI_1\beta^2)\,\frac{\partial^2 \alpha}{\partial x^2}+EI_0\beta\,\frac{\partial^2 h}{\partial x^2}-\rho I_0\,\frac{\partial^2 \alpha}{\partial t^2}=0 \qquad (1)$$

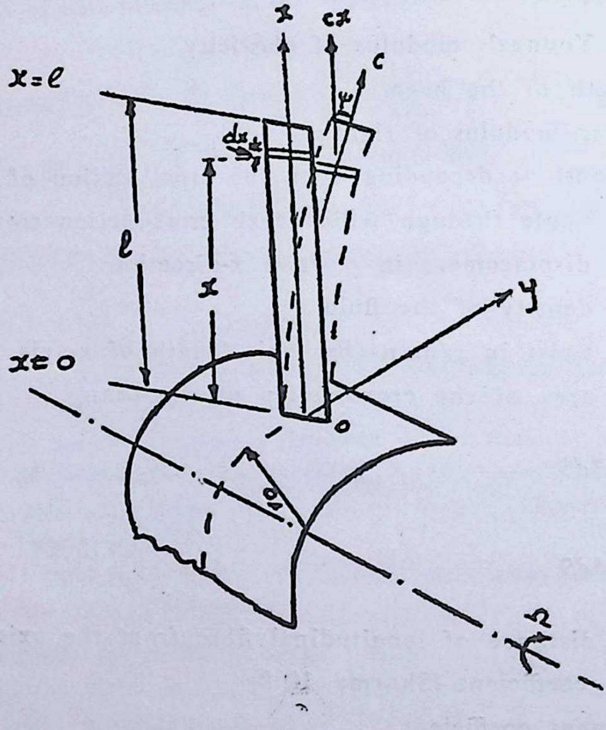


FIG. 1

When the elastic force is included the blade will be deformed. Also in a steady flow of speed U, the blade will be deformed due to the aerodynamic force. Then the above differential equations become

$$ES\frac{\partial^2 h}{\partial x^2}+EI_0\beta\frac{\partial^2\alpha}{\partial x^2}-\rho S\frac{\partial^2 h}{\partial t^2}+\frac{\rho U^2}{2}c\,C_L=0$$

$$(GJ+EI_1\beta^2)\frac{\partial^2\alpha}{\partial x^2}+EI_0\beta\frac{\partial^2 h}{\partial x^2}-\rho I_0\frac{\partial^2\alpha}{\partial t^2}+\frac{\rho U^2}{2}c^2\left[c_N+\frac{x_0}{c}c_L\right]=0. \tag{2}$$

The first of equations (2) represents equilibrium for axial force and, therefore, it is modified, if the centrifugal force (Sharma 1976) is taken into consideration. Then the governing differential equations are

$$ES\frac{\partial^2 h}{\partial x^2}+EI_0\beta\frac{\partial^2\alpha}{\partial x^2}-\rho S\frac{\partial^2 h}{\partial t^2}+\rho S\,\Omega^2\{r_0(l-x)+\tfrac{1}{2}(l^2-x^2)\}\times\frac{\partial h}{\partial x}+\frac{\rho U^2}{2}c\,C_L=0$$

$$(GJ+EI_1\beta^2)\frac{\partial^2\alpha}{\partial x^2}+EI_0\beta\frac{\partial^2 h}{\partial x^2}-\rho I_0\frac{\partial^2\alpha}{\partial t^2}+\frac{\rho U^2}{2}c^2\left[c_N+\frac{x_0}{c}c_L\right]=0. \tag{3}$$

The equations (3) are now put in terms of dimensionless variable by introducing $\xi=x/l$ and the substitutions

$$\alpha_0=\frac{E}{\rho l^2},\qquad K_2=\left(\frac{3}{4}-\frac{x_0}{c}\right)c,\qquad \nu=\Omega^2 l,$$

$$m=\frac{GJ+EI_1\beta^2}{\rho I_0 l^2},\qquad K_3=\frac{c^2}{2I_0}\,\frac{c_\pi}{\delta},\qquad \delta=\frac{EI_0\beta}{\rho S l^2},$$

$$K_1=\frac{c}{2S}\,\frac{dc_L}{d\alpha},\qquad K_4=\left(\frac{x_0}{c}-\frac{1}{4}\right)c,\qquad n=\frac{E\beta}{\rho l^2}.$$

The governing differential equation of the motion of the coupled torsional and longitudinal vibrations of a pretwisted slender beam under aerodynamic couplings in a steady flow of an incompressible fluid in a centrifugal force field is

$$\alpha_0\frac{\partial^2 h}{\partial\xi^2}+\nu\left\{\frac{r_0}{l}+\frac{1}{2}-\frac{r_0}{l}\xi-\tfrac{1}{2}\xi^2\right\}\frac{\partial h}{\partial\xi}-\frac{\partial^2 h}{\partial t^2}+\delta\frac{\partial^2\alpha}{\partial\xi^2}+K_1U^2\left\{\alpha+\frac{1}{U}\frac{\partial h}{\partial t}+\frac{K_2}{U}\frac{\partial\alpha}{\partial t}\right\}=0$$

$$n\frac{\partial^2 h}{\partial\xi^2}+m\frac{\partial^2\alpha}{\partial\xi^2}-\frac{\partial^2\alpha}{\partial t^2}-K_3U\frac{\partial\alpha}{\partial t}+K_1K_4\left[U^2\alpha+U\frac{\partial h}{\partial t}+UK_2\frac{\partial\alpha}{\partial t}\right]=0 \tag{4}$$

The boundary conditions are given by equations

$$h=\alpha=0\qquad\text{at}\qquad\xi=0$$

$$\frac{\partial h}{\partial\xi}=\frac{\partial\alpha}{\partial\xi}\qquad\text{at}\qquad\xi=1.$$

The solution can be expressed in the form

$$h(\xi, t) = A f(\xi)\, e^{\lambda t}$$

$$\alpha(\xi, t) = B\, \phi(\xi)\, e^{\lambda t} \tag{5}$$

where A, B, λ are constants; λ, $f(\xi)$ and $\phi(\xi)$ are to be obtained by the differential system and A, B by the initial condition. To obtain an approximate solution Galarkin's method has been used. We shall assume approximate forms of $f(\xi)$ and $\phi(\xi)$ to be known.

We substitute Equations (5) into Equations (4) and then multiply the first equation by f and the second by ϕ, integrate both the resulting Equations with respect to ξ from 0 to 1 to obtain two linear homogeneous equations for A and B. For a nontrivial solution corresponding to a disturbed motion, the determinant of coefficient of A and B must vanish.

$$\begin{vmatrix} -a_1+a_2-\lambda^2 a_3+U\lambda a_4 & -a_5+U^2 a_6+U\lambda a_7 \\ -a_8+U\lambda a_9 & -a_{10}-\lambda^2 a_{11}-U\lambda a_{12}+U^2 a_{13}+U\lambda a_{14} \end{vmatrix} = 0$$

It gives

$$A_1\lambda^4+A_2\lambda^3+A_3\lambda^2+A_4\lambda+A_5 = 0 \tag{6}$$

This determinantal equation is known as the characteristic equation where

$$a_1 = \alpha_0 \int_0^1 \frac{d^2 f}{d\xi^2} f\, d\xi$$

$$a_2 = \nu \int_0^1 \left(\frac{r_0}{l}+\frac{1}{2}-\frac{r_0}{l}\xi-\frac{1}{2}\xi^2\right)\frac{df}{d\xi}\, f\, d\xi$$

$$a_3 = \int_0^1 f^2 d\xi$$

$$a_4 = K_1 \int_0^1 f^2\, d\xi$$

$$a_5 = \delta \int_0^1 \frac{d^2\phi}{d\xi^2}\, f\, d\xi$$

$$a_6 = K_1 \int_0^1 f\phi\, d\xi$$

$$a_7 = K_2\, K_3 \int_0^1 f^2\, d\xi$$

$$a_8 = n \int_0^1 \frac{d^2 f}{d\xi^2}\, \phi\, d\xi$$

$$a_9 = K_1 K_4 \int_0^1 f\phi \, d\xi$$

$$a_{10} = m \int_0^1 \frac{d^2\phi}{d\xi^2} \phi \, d\xi$$

$$a_{11} = \int_0^1 \phi^2 \, d\xi$$

$$a_{12} = K_3 \int_0^1 \phi^2 d\xi$$

$$a_{13} = K_1 K_4 \int_0^1 \phi^3 \, d\xi$$

$$a_{14} = K_1 K_2 K_4 \int_0^1 \phi^3 \, d\xi$$

and

$$\begin{aligned}
A_1 &= a_3 a_{11} \\
A_2 &= [a_3 (a_{12} - a_{14}) - a_4 a_{11}] U \\
A_3 &= a_3 (a_{10} - a_{13}) - a_{11} (a_2 - a_1) + U^2 (a_4 a_{14} - a_7 a_9) \\
A_4 &= U [-a_4 a_{10} + a_{14} (a_2 - a_1) + a_7 a_8 + a_5 a_9] + U^3 (a_4 a_{13} - a_6 a_9) \\
A_5 &= a_{10} (a_1 - a_2) - a_5 a_8 + U^2 (a_2 - a_1) a_{13}.
\end{aligned} \tag{7}$$

The condition for stability of the disturbed motion is that the real parts of all the roots of the characteristic equation be negative (Fung 1955). It can be easily shown that the necessary and sufficient condition for this is that the coefficient A_1, A_2, A_3, A_4, A_5 and the Routh discriminant

$$R = A_2 A_3 A_4 - A_2^2 A_5 - A_1 A_4^2 \tag{8}$$

have the same sign. By Schwarz inequality the coefficients A_1 is always positive. Hence the conditions of stability become

$$A_2 > 0, \quad A_3 > 0, \quad A_4 > 0, \quad A_5 > 0, \quad R > 0 \tag{9}$$

These inequalities are satisfied when U is very small. As U increases, the coefficients A_2, A_3, A_4, A_5 and R will vary. Since $A_2 = [a_3 (a_{12} - a_{14}) - a_4 a_{11}] U$ is positive at all values of $U > 0$. As for A_3 and A_4, we shall show that A_5 and R become zero before A_2 and A_4. For if $A_4 = 0$, then

$$R = -A_2^3 A_5$$

which again shows that either A_5 or R must be negative. Similarly if $A_3 = 0$, then

$$R = -A_2^2 A_5 - A_4^2 A_1$$

which shows that either A_5 or R must be negative. Hence as long as A_5 and R remain positive, A_3 and A_4 must be positive. If A_5 and R remain positive, the motion will be stable. Alternately if either one becomes negative, the motion is unstable.

The divergence speed is given by

$$A_5 = E_1 + E_2\, U^2 = 0$$

where

$$E_1 = a_{10}\,(a_1 - a_2) - a_5\, a_8$$

$$E_2 = (a_2 - a_1)\, a_{13}$$

i.e.

$$U^2 \operatorname{div} = -E_1/E_2$$

The conditions $R = 0$ on the other hand, corresponding to flutter condition. Substituting A_1, A_2, A_3 etc. from equation (7) in (8) we get

$$R = U^2\,(PU^4 - QU^2 + R') = 0 \tag{10}$$

where

$$P = D_2\,(B_1\, C_2 - D_2\, A_1)$$

$$Q = -B_1\, C_2\, D_1 - B_1\, C_1\, D_2 + B_1^2\, E_2 + 2A_1\, D_1\, D_2$$

$$R' = D_1\,(B_1\, C_1 - A_1\, D_1) - B_1^2\, E_1$$

$$C_1 = a_3\,(a_{10} - a_{13}) - a_{11}\,(a_2 - a_1)$$

$$C_2 = a_4\, a_{14} - a_7\, a_9$$

$$D_1 = -a_4\, a_{10} + a_{14}\,(a_2 - a_1) + a_7\, a_8 + a_5\, a_9$$

$$D_2 = a_4\, a_{13} - a_6\, a_9$$

$$B_1 = a_3\,(a_{13} - a_{14}) - a_4\, a_{11}$$

From equation (10) we obtain the value of critical speed as given by

$$U^2 = \frac{Q \pm \sqrt{(Q^2 - 4PR')}}{2P}. \tag{11}$$

which is exactly the same as (13) (Sharma 1973). The equation (10) has a root $U = 0$ which gives that the structure may oscillate in still air. The special case arises when $R = 0$ at a certain higher value of U say U_{Cr} but becomes positive again at super critical speed which occurs only when R reaches a relative minimum at U_{Cr}. This is possible if and only if

$$R = \frac{dR}{dU} = 0, \qquad \frac{d^2R}{dU^2} > 0.$$

Using equation (10) it is seen that the dimensions and aerodynamic characteristic of the slender beam must be such that as to satisfy the conditions

$$Q^2 = 4PR^1 \quad \text{and} \quad Q < 0 \tag{12}$$

If this equation is satisfied at the critical nature then the critical nature of the motion at speed U_{Cr} is only superficial. At speeds differing slightly from

U_{Cr}, the disturbed motion is stable. The speed U_{Cr} is truly critical when equation (12) is not satisfied and for super critical speeds the motion will diverge.

Numerical Example : A numerical example for the stability of the disturbed motion of a twisted strip under centrifugal and aerodynamic force field has been considered. The cross-section of the beam is taken as rectangle having width $2b$ and depth $2h$. The formula $J \sim 4k_1 Sh^3$ is used to calculate J, where $K_1 = 0.312$ for $h/b = 1/10$ and $1/3$ for $h/b = 1/15$ and $1/20$ (Timoshenko and Goodier 1951). The other physical constants of the beam are taken as given below :

$$E = 30 \times 10^6 \text{ lbs./in}^2, \; G = 11.25 \times 10^6 \text{ lbs./in}^2$$

$$r_0 = 32 \text{ in}, \; \Omega = 314 \text{ Sec}^{-1}, \; l = 8 \text{ in}, \; b = 1 \text{ in}, \; c = 0.24885 \text{ in}$$

$$\rho = 0.00082 \text{ lbs/in}^3, \; dc_L/d\alpha = 6 \text{ radians} < 2\pi.$$

With these values the various parameters used to find the values of A_1, A_2, A_3, A_4, A_5 are as follows :

$$a_1 = 30.487768 \times 10^7$$
$$a_2 = 0.021722 \times 10^7$$
$$a_3 = 0.120635$$
$$a_4 = 0.034363$$
$$a_5 = 1.102307 \times 10^7$$
$$a_6 = 0.066734$$
$$a_7 = 0.051644$$
$$a_8 = 0.330704 \times 10^8$$
$$a_9 = 0.055455$$
$$a_{10} = 1.414446 \times 10^7$$
$$a_{11} = 0.5$$
$$a_{12} = 0.335720$$
$$a_{13} = 0.118353$$
$$a_{14} = 0.091592$$

Quantities A_1, A_5 and R were computed and found to be positive. The motion was thus stable. The critical speed U_{Cr} was computed and calculations indicated that the condition (12) was not fulfilled. It was therefore, concluded that the determined value U_{Cr} was truly critical.

References

Fung, F. C (1955) : An Introduction to the Theory of Aeroelasticity. John Wiley and Sons Inc : New York.

Sharma, B. B. (1976) : Coupled Torsional and Longitudinal Vibrations of a Pretwisted Slender beam under Aerodynamic Coupling in Centrifugal Force Field. *Journal of Pure and Applied Mathematics.* 7, 4.

Sharma, B. B. (1973) : The Stability of Disturbed Motion of Coupled Torsional Vibrations of a Slender Beam Involving Inertia, Elastic snd Aerodynamic Coupling. *Mathematics Education.* VII, 1.

Sharma, B. B. (1978) : Coupled and Torsional Vibrations of a Slender beam under Aerodynamic Coupling in Centrifugal Force Field (Accepted) *Bull. Cal.. Math, Soc.*

Timoshenko and Goodier, J. N. (1951) : Theory of Elasticity. McGraw Hill Book Co. Inc. New York.

DEPARTMENT OF MATHEMATICS,
H. P. UNIVERSITY, SIMLA, (INDIA).

Bull. Cal. Math. Soc., 75, 239—243 (1983)

TWO PARAMETER LIE GROUPS AND SPECIAL FUNCTIONS

S. K. Chatterjea and T. D. Banerjee

(*Received 20 April 1981*)

1. Introduction : The group theoretic study of Bessel Function of the first kind was made by B. Kaufmen (1966) by means of the relations.

$$R\left(\exp(in\varphi)\, J_n(r)\right) = \exp\left(i(n+1)\right)\varphi\, J_{n+1}(r) \tag{1.1}$$

$$L\left(\exp(in)\, J_n(r)\right) = \exp\left(i(n-1)\right)\varphi\, J_{n-1}(r). \tag{1.2}$$

where

$$R = \exp(i\varphi)\left(-\frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial r}\right),$$

$$L = \exp(-i\varphi)\left(-\frac{i}{r}\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial r}\right),$$

and

$$[R, L] = RL - LR \equiv 0, \quad L \not\equiv P(r, \varphi)\, R,$$

In connection with simple Bessel polynomials the first author (Kaufman 1966; Chatterjea 1978) considered a two-parameter Lie group by means of the relations

$$R\left(\exp\left(i(n+1/2)\right)\varphi\, Z_n(r) = \exp\left(i\,(2n+3/2)\varphi\right) Z_{n+1}(r) \tag{1.3}$$

$$L\left(\exp\left(i(n+1/2)\right)\varphi\, Z_n(r) = \exp\left(i\,(2n-1/2)\varphi\right) Z_{n-1}(r) \tag{1.4}$$

where

$$R = \exp(i\varphi)\left(-\frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial r}\right),$$

$$L = \exp(-i\varphi)\left(\frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{\partial}{\partial r}\right)$$

and

$$Z_n(r) = r^{-1/2} e^{-r} y_n\left(\frac{i}{r}\right),$$

$$[R, L] \equiv 0, \; L \not\equiv \rho\,(r, \varphi)\, R$$

The object of the present paper is to point out that the study of the above two-parameter groups can be much more simplified if one uses Lie's canonical form of such group.

For this purpose, let us consider, in general, the following two operators,

$$R=\xi_1\,(x,y)\,\frac{\partial}{\partial x}+\eta_1\,\frac{\partial}{\partial y}+\lambda_1\,(x,y) \tag{1.5}$$

$$=\xi_2\,(x,y)\,\frac{\partial}{\partial x}+\eta_2\,(x,y)\,\frac{\partial}{\partial y}+\lambda_2\,(x,y) \tag{1.6}$$

such that $[RL]\equiv 0$ and $L\not\equiv p\,(x,y)\,R$,

According to Lie, a set of variables $\bar{x}$, $\bar{y}$ can be determined so that takes the form

$$R\equiv\frac{\partial}{\partial\bar{x}}\;.$$

If then the resulting form of L is

$$L\equiv\bar{\xi}_2\,\frac{\partial}{\partial\bar{x}}+\bar{\eta}_2\frac{\partial}{\partial\bar{y}}+\lambda_2(\bar{x},\bar{y}),$$

then

$$[R,L]=\frac{\partial\bar{\xi}_2}{\partial\bar{x}}\cdot\frac{\partial f}{\partial\bar{x}}+\frac{\partial\bar{\eta}_2}{\partial\bar{x}}\cdot\frac{\partial f}{\partial\bar{y}}+\frac{\partial\bar{\lambda}_2}{\partial\bar{x}}\;f.$$

But $[R,L]\equiv 0$. Thus we have

$$\frac{\partial\bar{\xi}_2}{\partial\bar{x}}=0,\quad\frac{\partial\bar{\eta}_2}{\partial\bar{x}}=0,\quad\frac{\partial\bar{\lambda}_2}{\partial\bar{x}}=0,$$

so that $\bar{\xi}_2$ $\bar{\mu}_2$ and $\bar{\lambda}_2$ are at most functions of $\bar{y}$.

Let us put

$$L=\xi_2(\bar{y})\,\frac{\partial}{\partial\bar{x}}+\bar{\eta}_2\,(\bar{y})\,\frac{\partial}{\partial\bar{y}}+\bar{\lambda}_2\,(\bar{y})$$

Now the operator $R=(\partial/\partial\bar{x})$ remains unchanged due to change of variables of the type

$$X=\bar{x}-\phi\,(\bar{y})$$
$$Y=\psi\,(\bar{Y}),$$

where $\varphi\,(\bar{y})$ and $\psi\,(\bar{y})$ are at our disposal.

This change of variables causes Lf to take the form

$$[\xi_2\,(\bar{y})-\bar{\eta}_2\,(\bar{y})\,\varphi'\,(\bar{y})]\,\frac{\partial f}{\partial X}+\bar{\eta}_2\,(\bar{y})\,\psi'\,(\bar{y})\,\frac{\partial f}{\partial Y}+\bar{\lambda}_2\,(\bar{y})\,f$$

If we choose

$$\varphi=\int\frac{\bar{\xi}_2(\bar{y})}{\bar{\eta}_2(\bar{y})}\,d\bar{y}\quad\text{and}\quad\psi=\int\frac{d\bar{y}}{\bar{\eta}(\bar{y})}$$

L assumes the form

$$\frac{\partial}{\partial Y}+\bar{\lambda}_2\,(\bar{y})$$

In particular, if $\lambda_2(x, y) \equiv 0$, then α assumes the form $\partial/\partial y$. On the otherhand, the operators (1.5) and (1.6) can be reduced to the forms

$$R = \frac{\partial}{\partial X}, \qquad L = \frac{\partial}{\partial Y},$$

provided X satisfies

$$\begin{aligned} \xi_1 \frac{\partial X}{\partial x} + \eta_1 \frac{\partial X}{\partial y} + \lambda_1 X &= 1 \\ \xi_2 \frac{\partial X}{\partial x} + \eta_2 \frac{\partial X}{\partial y} + \lambda_2 X &= 0 \end{aligned} \tag{1.7}$$

and Y satisfies

$$\begin{aligned} \xi_1 \frac{\partial Y}{\partial x} + \eta_1 \frac{\partial Y}{\partial y} + \lambda_1 Y &= 0 \\ \xi_2 \frac{\partial Y}{\partial x} + \eta_2 \frac{\partial Y}{\partial y} + \lambda_2 Y &= 1. \end{aligned} \tag{1.8}$$

In particular if $\lambda_1(x, y) \equiv 0$ and $\lambda_2(x, y) \equiv 0$ then (1.7) and (1.8) reduce to the form

$$\begin{aligned} \xi_1 \frac{\partial X}{\partial x} + \eta_1 \frac{\partial X}{\partial y} &= 1 \\ \xi_2 \frac{\partial X}{\partial x} + \eta_2 \frac{\partial X}{\partial y} &= 0 \end{aligned} \tag{1.9}$$

$$\begin{aligned} \xi_1 \frac{\partial Y}{\partial x} + \eta_1 \frac{\partial Y}{\partial y} &= 0 \\ \xi_2 \frac{\partial Y}{\partial x} + \eta_2 \frac{\partial Y}{\partial y} &= 1. \end{aligned} \tag{1.10}$$

2. Bessel Function of the first kind : In (Kaufman 1966) we notice that $R\{e^{in\phi} J_n(r) = e^{i(n+1)\phi} J_{n+1}(r)$ and $L(e^{in\phi} J_n(r)) = e^{i(n-1)\phi} J_{n-1}(r)$ where

$$R = e^{i\phi}\left(-\frac{i}{r}\frac{\partial}{\partial\phi} - \frac{\partial}{\partial r}\right)$$

$$L = e^{-i\phi}\left(-\frac{i}{r}\frac{\partial}{\partial\phi} + \frac{\partial}{\partial r}\right).$$

Evidently $[RL] \equiv 0$ and $L \not\equiv (r, \phi) R$.

Thus using the formula (1.1) to (1.4) we obtain the following canonical variables

$$\Phi = -\frac{r}{2} e^{-i\phi}, \qquad R = \frac{r}{2} e^{i\phi}, \tag{2.1}$$

so that

$$R \equiv \frac{\partial}{\partial \Phi}, \qquad L \equiv \frac{\partial}{\partial R}. \tag{2.2}$$

We note that

$$r = 2\,(-\Phi R)^{1/2}$$

$$e^{i\phi} = (-R/\Phi)^{1/2} \tag{2.3}$$

If we put

$$F_n\,(\Phi,\ R) = \left(-\frac{R}{\Phi}\right)^{n/2}\ J_n\,(2\,\sqrt{\{-\Phi R\}}\,)$$

we have on using canonical form (2.2),

$$e^{\alpha R}\,F_n\,(\Phi,\ R) = F_n\,(\Phi+\alpha,\ R).$$

On the otherhand

$$e^{\alpha R}\,F_n\,(\Phi,\ R) = \sum_{m=0}^{\infty}\ \frac{\alpha^m}{m\,!}\ F_{n+m}\,(\Phi,\ R)$$

Equating these two results and transacting back to original variables we get,

$$\left(1-\frac{2\alpha}{r}\ e^{i\phi}\right)^{-n/2} J_n\,(\sqrt{\{r^2-2\alpha r\,e^{i\phi}\}}\,) = \sum_{m=0}^{\infty}\ \frac{(\alpha\,e^{i\phi})^m}{m\,!}\ J_{n+m}(r), \tag{2.4}$$

which can be written in elegant form

$$\left(1-\ \frac{2\rho}{r}\right)^{-2/2} J_n\ \sqrt{(r^2-2r\rho)} = \sum_{m=0}^{\infty}\ \frac{\rho^m}{m\,!}\ J_{n+m}(r). \tag{2.5}$$

Similarly applying the operator

$$e^{\alpha L} \qquad \text{on} \qquad F_n\,(\Phi,\ R),$$

we obtain

$$\left(1+\frac{2\rho}{r}\right)^{n/2}\ J_n\,(\sqrt{\{\,r^2+2r\rho)} = \sum_{m=0}^{\infty}\ \frac{\rho^m}{m\,!} J_{n-m}(r) \tag{2.6}$$

3. Bessel Polynomials : In (Chatterjea 1978) we notice that

$$R\,(e^{i\,(x+1/2)\phi}\,Z_n(r)\,) = e^{(i\,(2\,n+2)/3)\phi}\ Z_{n+1}(r)$$

and

$$L\,(e^{i\,(x+1/2)\phi}\,Z_n(r)\,) = e^{i\,(2n-1)/2)\phi}\ Z_{n-1}(r)$$

where

$$R \equiv\ e^{i\phi}\left(-\frac{i}{r}\ \frac{\partial}{\partial\phi} - \frac{\partial}{\partial r}\right)$$

$$L \equiv\ e^{-i\phi}\left(\ \frac{i}{r}\ \frac{\partial}{\partial\phi} - \frac{\partial}{\partial r}\right)$$

and $Z_n(r) = r^{-1/2}\,e^{-r}\,y_n\,(1/r)$ and $y_n\,(1/r)$ is the Bessel Polynomial satisfying the differential equation

$$x^2\ \frac{d^2u}{dx^2}+2(x+1)\ \frac{du}{dx} = n(n+1)\,u.$$

Evidently

$$[\,R,\ L\,] \equiv 0 \qquad \text{and} \qquad L \not\equiv \rho(r,\ \phi)\ R.$$

So using the formula (1.1)—(1.4) we obtain the following canonical variables

$$\Phi = -\frac{r}{2}\ e^{-i\phi}, \qquad R = -\frac{r}{2}\ e^{i\phi} \tag{3.1}$$

so that

$$R \equiv \frac{\partial}{\partial \Phi}, \qquad L \equiv \frac{\partial}{\partial R}. \tag{3.2}$$

We note that

$$r = 2\,(\Phi R)^{1/2}, \qquad e^{i\phi} = (R/\Phi)^{1/2}. \tag{3.3}$$

If we put,

$$F_n\,(\Phi, R) = (4\Phi R)^{-1/4}\left(\frac{R}{\Phi}\right)^{2n+1/6} \exp\left(-2\sqrt{(\Phi R)}\,\right) y_n\left(\tfrac{1}{2}\sqrt{(1/\Phi R)}\,\right).$$

We have after applying the operator $e^{\alpha R}$ upon $F_n\,(\Phi.\ R)$ and then translating back to the original variables

$$(1-(2\alpha/r)\,e^{i\phi})^{-(1/2)(n+1)} \exp\{\,r-\sqrt{(r^2-2\alpha r\,e^{i\phi})}\,\}.$$

$$.\ y_n\left(\frac{1}{\sqrt{(r^2-2\alpha r\,e^{i\phi})}}\right) = \sum_{m=0}^{\infty} \frac{L^m}{m\,!}\ e^{im\phi}\ y_{n+m}\ (1/r) \tag{3.4}$$

which can be written in elegant form

$$\left(1-\frac{2\rho}{r}\right)^{-(1/2)(n+1)} \exp\left\{\,r-\sqrt{(r^2-2\rho r)}\,\right\} y_n\left(\frac{1}{\sqrt{(r^2-2\rho r)}}\right)$$

$$= \sum_{m=0}^{\infty} \frac{\rho^m}{m\,!}\ y_{n+m}\left(\frac{1}{r}\right). \tag{3.5}$$

Similarly applying the operator $e^{\alpha L}$ on $F_n\ (\Phi, R)$ we obtain

$$\left(1-\frac{2\rho}{r}\right)^{n/2} \exp\left\{\,r-\sqrt{(r^2-2\rho r)}\,\right\} y_n\left\{\frac{1}{\sqrt{(r^2-2\rho r)}}\right\}$$

$$= \sum_{m=0}^{\infty} \frac{\rho^m}{m\,!}\ y_{n-m}\left(\frac{1}{r}\right), \tag{3.6}$$

where it should be remembered that definition of Bessel polynomials can be extended to negative subscripts by defining $y_{-n}(x)$ to $y_{n-1}(x)$. It may be noted that the formulae (3.5) and (3.6) are analogous to (2.5) and (2.6) respectively.

References

Chatterjea, S. K. (1978) : *C. R. Acad. Sc. Paris*, 271, Sers, 357.

Chatterjea. S. K. : Some properties on simple Bessel polynomials from the view point of Algebra,

Kaufman, S. (1966) : *J. Math. Physics*, 7, 447.

Weisner, L. (1959) : *Canad. Jr. Math.* II, 148.

DEPARTMENT OF PHYSICS
PRESIDENCY COLLEGE,
CALCUTTA-700 073, INDIA.

Bull. Cal. Math. Soc., 75, 244—248 (1984)

FIXED POINT THEOREMS IN LOCALLY UNIFORM CONVEX BANACH SPACE

N. P. S. BAWA and R. C. VARMA

(*Received 12 July 1979*)

Abstract : In this joint paper the fixed point for non-expansive mapping acting on a weakly closed, bounded and convex subset as well as weakly compact convex subset of locally uniform convex Banach space with certain restriction on the mapping has been investigated.

1. **Introduction** : Just as Banach's Contraction Principle is the basis to prove the existence of fixed point for any map in a metric space similarly the theorems due to Schauder (1930) and Tychnoff (1935) are the backbone to ensure fixed point for non-expansive mapping in the domain of normed linear spaces or in general, locally convex Hausdorff linear topological spaces. Krasnoselskii (1955) has proved that a sequence of iterates of mapping F acting on a closed and convex subset of uniform convex space converges to the fixed point of the non-expansive mapping. Edelstein (1961) has remarked that the above assumption of uniform convexity is unnecessary in case of Krasnoselskii (1955) and mere strict convexity will suffice to prove the result. Browder (1965) has asserted the condition of boundedness in addition to closed and convex one for the subset of uniform convex Banach space. There he has shown that any self non-expansive mapping on this subset has a fixed point. Browder (1968) has constructed number of new types of linear or non-linear operators depending upon the boundary conditions he happens to encounter in the field of differential and integral equations. He has applied them in various forms in different types of Banach spaces to show the existence of fixed point in such spaces. Using the concept of minimax inequality Ky-Fan (1962) has proved the number of results due to Tychnoff (1935) for the locally convex linear topological spaces without the condition of Hausdorff. But in the literature on fixed point we have so far not found any theorem where there is the localised form of the uniform convex Banach spaces.

In this paper, therefore, we have tried to establish two theorems on fixed point for the self non-expansive mapping in the locally uniform convex Banach space. In the first one a weakly closed bounded and convex subset of locally uniform convex Banach space has been considered. In the second case a weakly compact with convexity condition subset of the same space has been taken. In the later case the result is proved for weakly continuous non-expansive mapping

2. We will require following notations, definitions and results.

Throughout this paper, we will use complete arrow $\rightarrow$ to denote the strong convergence and the upper half arrow $\rightharpoonup$ to denote the weak convergence.

The definition of locally uniform convex Banach space in term of cnnvergence as given in Hewitt and Stromberg (1969) as follows.

Definition : *A Banach space X over the real or complex is said to be locally uniform convex, if for a sequence* $<x_n>$ *in X and* $x \in X$ *we have*

$$\| x_n \| = 1 \text{ and } \| x \| = 1, \text{ then the relation}$$

$$\left\| \frac{x_n + x}{2} \right\| \rightarrow 1$$

implies

$$\| x_n - x \| \rightarrow 0. \tag{1}$$

We know that the convergence may be weak or strong depending upon whether the convergence is in the mean or not. The condition for the weak oonvergence in the locally uniform convex Banach space can be best ascertained by the following condition.

2.2. Definition : *Let X be a locally uniform convex Banach space. The sequence* y_n *of elements of X converges to an element y of X if the following condition holds,*

$$\text{(i)} \quad \| y_n \| \rightarrow \| y \|$$

$$\text{(ii)} \quad y_n \rightharpoonup y \quad \textit{weakly which}$$

implies

$$\| y_n - y \| \rightarrow 0. \tag{2}$$

In order to convert weak convergence into strong convergence we make use of demiclosed mapping which is defined as follows.

2.3 Definition : *Let X be a Banach space and C be its subset. A mapping* $T: C \rightarrow X$ *is said to be demiclosed provided that*

$$x_n \in C, \text{ and } x_n \rightharpoonup x \in C \Rightarrow Tx_n \rightarrow y \in X. \tag{3}$$

Then $Tx = y$.

We shall now define the non-expansive mapping in Banach space.

2.4. Definition : *A self mapping T is said to be non-expansive in Banach space X, provided*

$$\| Tx - Ty \| \leqq \| x - y \| \tag{4}$$

holds for all x, y in X.

We shall require the following results proved in Kelley (1968).

2.5. PROPOSITION : *Every weakly compact subset of locally convex space is complete.*

The most striking result which is frequently used for the weak topology in a linear topological space is the Eberlein-Smulian theorem as given in Kelley (1968).

2.6. THEOREM: *Every weakly closed convex subset of a weakly compact set in a complete locally convex space is weakly sequentially compact.*

3. We prove the following.

3.1. THEOREM: *Suppose C is weakly closed, bounded and convex subset of locally uniform convex Banach space X. Let T be self non-expansive mapping of C into C such that $I-T$ is demiclosed, then T has a fixed point in C.*

Proof: We know that weakly closed bounded subset of Banach space is weakly compact, therefore, by the proposition 2.5 it is complete. By Eberlein-Smulian theorem 2.6 C is also a weakly sequentially compact, so there is a subsequence $<x_{n_j}>$ of $<x_n>$ such that $x_{n_j} \rightharpoonup x$ with $\| x_{n_j} \| = 1$ and $\| x \| = 1$. As C is convex, defining $T_f : C \to C$.

$$T_f(x) = T(x) + t\, f(x) \text{ for } t \in (0\ 1) \text{ and } f \in X^x \text{—a congugate of } x$$

where

$$\| f \| = 1. \tag{5}$$

We have to show that $T_f(x_{n_j}) \rightharpoonup T_f(x)$ weakly and T_f is non-expansive. Now

$$\| T_f(x_{n_j}) - T_f(x) \| \leq \| Tx_{n_j} - Tx \| + t \| f(x_{n_j}) - f(x) \| \tag{5A}$$

Since C is weakly compact, and the functionals corresponding to $<x_n>$ converges uniformly in C, $\Rightarrow \| f(x_{n_j}) - f(x) \| \to 0$. Moreover, as T is non-expansive the above relation (SA) reduces to

$$\| T_f(x_{n_j}) - T_f(x) \| \leqslant \| x_{n_j} - x \| \tag{6}$$

This shows that T_f is also non-expansive.

Again, $x_{n_j} \rightharpoonup x$ and T is continuous, therefore, $T_f(x_{n_j}) \rightharpoonup T_f(x)$. T_f being non-expansive, and continuous so $T_j(x_{n_j})$ will have uniform convergence in C. Thus we have

$$\| T_f(x_{n_j}) \| \to \| T_f(x) \|$$

Hence by definition 2.2, it yields

$$\| T_f(x_{n_j}) - T_f(x) \| \to 0 \tag{7}$$

As $I-T$ is demiclosed, we have

$$\begin{aligned} \| I(x) - T(x) \| &= \| I_f(x) - T_j(x) \| \\ &\leq \| x - x_{n_j} + x_{n_j} - T_f(x_{n_j}) + T_f(x_{n_j}) - T_f(x) \| \\ &\leq \| x - x_{n_j} \| + \| x_{n_j} - T_f(x_{n_j}) \| + \| T_f(x_{n_j}) - T_f(x) \| . \end{aligned}$$

By using condition (7) and definition 2.2. it implies

$$\| I(x) - T(x) \| \to 0 \text{ strongly.} \tag{8}$$

Thus $T(x)=x$ as X is Hausdorff.

This completes the proof.

To prove the next theorem we define the concept of weakly jointly continuous function. A family $F=\{f_\alpha\}$ of functions from $[0, 1]$ into a subset S of Banach space will be called jointly weakly continuous provided $t \to t_0$ in $[0, 1]$ and $\alpha \to \alpha_0$ in S, then $f_\alpha(t) \to f_{\alpha_0}(t_0)$ in S. Based on this idea we have tried to prove the following theorem.

3.2 THEOREM : *Suppose C is weakly compact convex subset of locally uniform convex Banach space X and suppose there exists a contraction jointly weakly family F of functions associated with C as described above. Then any non-expansive weakly continuous self mapping T of C has a fixed point in C.*

Proof : To prove the above theorem we first impose some conditions on $\{f_\alpha\}$ to make them as contraction. We define

$$f_\alpha(I)=\alpha \quad \text{for each} \quad \alpha \in C. \tag{9}$$

We also take a function $\varphi : (0, I) \to (0, I)$ such that for $\alpha, \beta \in C$ and $t \in (0, I)$, we have

$$\| f_\alpha(t)-f_\beta(t) \| \leq \varphi(t) \| \alpha-\beta \| . \tag{10}$$

Under this condition F is said to be contractive. Let $K_n = n/n+I$. Considering a map,

$$T_n : C \to C \text{ such that } T_n(x)=f_{T(x)}(k_n) \;\forall\; x \in C,$$

we see that T is well defined and maps C into C. Furthermore, for each n, and for all x, y in C, we have

$$\begin{aligned} \| T_n x-T_n y \| &= \| f_{Tx}(k_n)-f_{Ty}(k_n) \| \\ &\leq \varphi(k_n) \| Tx-Ty \| \\ &\leq \varphi(k_n) \| x-y \| \end{aligned} \tag{11}$$

so that for each n, T_n is a contraction.

Now by Eberlein-Smulian theorem, C is weakly sequentially compact. Thus there is a subsequence $\langle x_{n_j} \rangle$ of $\langle x_n \rangle$ such that $x_{n_j} \to x$. Now from (11), we get

$$\| T_{n_j}(x_{n_j})-T_n(x) \| \leq \varphi(k_n) \| x_{n_j}-x \|$$

$$\Rightarrow T_{n_j}(x_{n_j}) \rightharpoonup T_n(x) \quad \text{weakly}.$$

Since the weak topology of X is Hausdorff and C is weakly closed, therefore it is strongly closed.

So therefore

$$T_{n_j}(x_{n_j}) \to T_n(x).$$

Again, C is also complete as metric space with the normed topology of X, so each T_{n_j} has unique fixed point, that is, $T_{n_j}(x_{n_j}) = x_{n_j} \Rightarrow T_{n_j}(x_{n_j}) \to x$. The joint continuity of mappings now yields

$$T_{n_j}(x_{n_j}) = f_{Tx_{n_j}}(k_{n_j}) \to f_{Tx}(I) = T(x).$$

It follows $T(x)=x$, since X is Hausdorff.

Hence the theorem.

References

Browder, F. E. (1965): *Proc. Nat. Acad. Sci. U.S.A* **54**, 1041.

— — (1968): *Bull. Amer. Math. Soc.* **74**, 660.

Edelstein, M. (1961): *Proc. Amer. Math. Soc.* **12**, 7.

Fan, Ky. (1962): *Proc. Nat. Acad. Sci. U.S.A.* **38**.

Kelley, J. L. and Nimioka, I. (1968): Linear Topological Spaces, East-West Ed. D.-Van Nastrand Co. Inc., 153.

Krasnoselskii, M. A. (1955): *Uspehl, Math. Nauk.* 163, 123.

Scauder, J. (1930): *Studia Math.* 170.

Stromberg, K. and Hewitt, E. (1969): Real and Abstract Analysis, ***Springer-Verlang, New-York***, **232.**

Tychnoff, A. (1935): *Math. Ann.* 111.

DEPARTMENT OF MATHEMATICS
GOVERNMENT SCIENCE COLLEGE
REWA (M.P.)

DEPARTMENT OF MATHEMATICS
GOVERNMENT ENGINEERING COLLEGE
JABALPUR (M.P.)

Bull. Cal. Math. Soc., 75, 249—256 (1983)

ON WAVE PROPAGATION IN RANDOM MAGNETO-VISCO-ELASTIC MEDIUM

RABINDRA KUMAR BHATTACHARYYA

(*Received 2 September 1980. Revision Received 24 August 1981*)

Abstract : The problem of wave propagation in an infinite, interacting random "weakly conducting" magneto-visco-elastic medium has been considered. The random medium is assumed to differ slightly from the homogeneous medium. The dispersion equation has been derived and analysed to show that upto the first order the effect of randomness is to alter the wave-velocity and the attenuation coefficient ; the decrease in attenuation coefficient is proportional to the random fluctuation of the conductivity of the medium and the square of the corresponding magnetic induction.

Introduction : The interaction of magnetic, thermal and elastic or visco-elastic fields in an infinite non-random medium has been under study for some time. Knopoff (1955) studied the effect of presence of magnetic field in elastic wave propagation. Paria (1962) and Willson (1963) have discussed the propagation of plane waves in a magneto-thermo-elastic interacting field. Following Lessen (1957), Sinha (1964) investigated the case of plane wave propagation in a thermo-visco-elastic medium permeated by a magnetic field.

However, the study of random fields, especially in continuous media, is of recent origin. A general review of physical situations governed by random differential equations has been given by Van Kampen (1976). Karal and Keller (1964) examined the problem of wave propagation in random elastic and electro-magnetic media by using the method of smooth perturbation. Following the theory of Karal and Keller, Chow (1973) treated the propagation in a random thermo-elastic medium and Bhattacharyya (1979) considered the propagation of waves in a random thermoelastic non-simple medium.

In the present paper also, the formulation of Karal and Keller has been employed to investigate the case of magneto-visco-elastic waves in a random medium. The field equations have been put in the form

$$Lv = f \tag{1}$$

where L is a random linear differential operator, v the field vector and f represents a given non-random source. Then assuming

$$L = L_0 + \epsilon L_1 + \epsilon^2 L_2 \tag{2}$$

where L_0 and L_1, L_2 are constant and randomly fluctuating parts of L respectively and ϵ is a small parameter measuring the scale of random fluctuation of inhomogeneities of the medium, it can be shown that the mean field $< v >$ is governed by the equation

$$[\,L_0-\epsilon<L_1>-\epsilon^2<L_2>+\epsilon^2\,(<L_1>L_0^{-1}<L_1>-<L_1L_0^{-1}L_1>)\,]<v>\,=f \tag{3}$$

where

$$L_0\,G_{ij}\,(x,x',t,t')=\delta(x,x')\,\delta(t,t')\,\delta_{ij} \tag{4}$$

G_{ij} being the appropriate Green's tensor on L_0. The dispersion equation for waves propagating in the interacting randomly varying magneto-visco-elastic field is then deduced. It has been shown that the dispersion equation determines two types of wave velocities coupled with one another and that the attenuation coefficient of the 'mainly longitudinal' and 'mainly transverse' type of waves decreases by an amount proportional to $\sigma_1'\,v_0^2\,H_{02}^2$ and $\sigma_1'\,v_0^2\,H_{01}^2$ respectively, where v_0 is the non-fluctuating part of magnetic permeability, Ho_1 and Ho_2 are the longitudinal and transverse components of the primary magnetic field $\vec{H}_0$ and σ_1' is the mean of the fluctuating part of the electrical conductivity of the medium.

It may be pointed out that unlike Karal and Keller (1964) and Chow (1973) the effect of random fluctuation has been obtained in the first order terms involving the perturbation parameter ϵ; this has led to a considerable simplification in the calculations since only the second order terms involve the Green's tensor corresponding to the magneto-visco-elastic medium. Here we have assumed the medium to be weakly conducting, the conductivity has only a non-zero random part of the first order in the perturbation parameter. Chow has assumed a similar form for the thermo-elastic coupling parameter. If the non-random part of the conductivity is assumed to be non-zero, the effect of randomness is observed only in the second order terms; this case is being studied separately.

Problem : Let $\vec{u}$ be the displacement vector in a random magneto-visco-elastic medium, $\vec{H}$ the magnetic vector and $\vec{j}$ the current density vector of the electromagnetic field, and v the magnetic permeability. Then the equation of motion in the interacting field is given by

$$\sigma_{ij,j}+v(\vec{j}\times\vec{H})_i=\rho\ddot{u}_i \tag{5}$$

where

$$\begin{aligned}\sigma_{ij}&=S_{ij}+\tfrac{1}{3}\,\sigma_{kk}\,\delta_{ij}\\ e_{ij}&=\epsilon_{ij}-\tfrac{1}{3}\,\epsilon_{kk}\,\delta_{ij}\\ \epsilon_{ij}&=\tfrac{1}{2}\,(u_{i,j}+u_{j,i})\end{aligned} \tag{6}$$

S_{ij} = deviatoric component of stress

e_{ij} = deviatoric component of strain.

We set

$$(\sigma_{ij},\epsilon_{ij},u_j,\,\dots)=Re\,[\,(\bar{\sigma}_{ij},\bar{\epsilon}_{ij},\bar{u}_j,\,\dots)\exp(-i\omega t)\,] \tag{7}$$

where $\bar{\sigma}_{lj}$, $\bar{\epsilon}_{lj}$ etc. are functions, in general complex, of spatial co-ordinates only. We assume

$$\bar{S}_{jj} = Y_s\,\bar{e}_{lj}, \qquad \bar{\sigma}_{kk} = Y_v\,\bar{\epsilon}_{kk} \tag{8}$$

where Y_s and Y_v are the deviatoric and dilatational complex moduli respectively. Also setting

$$\bar{\lambda} = \tfrac{1}{3}(Y_v - Y_s), \qquad \bar{\mu} = \tfrac{1}{2}\,Y_s \tag{9}$$

we obtain

$$\bar{\sigma}_{lj} = \bar{\lambda}\,\delta_{lj}\,u_{k,k} + \bar{\mu}\,(\bar{u}_{l,j} + \bar{u}_{j,l})$$

Then it can be shown that

$$\begin{aligned}\bar{\sigma}_{lj,j} = [\,(\bar{\lambda}+\bar{\mu})\,\vec{\nabla}\,(\vec{\nabla}\,.\,) + \mu\nabla^2 + (\vec{\nabla}\,\bar{\lambda})\,(\vec{\nabla}\,.\,) \\ + \vec{\nabla}\,\bar{\mu}\times(\vec{\nabla}\times) + 2\,(\vec{\nabla}\,\bar{\mu}\,.\,\vec{\nabla})\,]\,\vec{\bar{u}}\end{aligned} \tag{10}$$

Also the electromagnetic field is governed by the equations (expressed in M. K. S. units):

$$\begin{aligned}&(i)\ \ \text{Curl}\ \vec{h} = -\vec{j}, \qquad \vec{H} = \vec{H}_0 + \vec{h} \\ &(ii)\ \ \text{Curl}\ \vec{E} = -\nu\,(\partial\vec{h}/\partial t), \qquad (iii)\ \ \text{div}\ (\nu\vec{H}) = 0 \\ &(iv)\ \ \vec{j} = \sigma\,[\,\vec{E} + \nu\,(\partial\vec{u}/\partial t\times\vec{H})\,]\end{aligned} \tag{11}$$

where $\vec{H}_0$ is the constant primary magnetic field, $\vec{h}$ the perturbed magnetic field and σ the electrical conductivity of the medium.

Neglecting small quantities of second order such as $\partial\vec{u}/\partial t\times\vec{h}$, the magnetic field can be shown to be dertermined by the equation

$$\begin{aligned}\nu\frac{\partial\vec{h}}{\partial t} = \frac{1}{\sigma}\Big[\vec{\nabla}\Big\{\frac{1}{\nu}\Big((\vec{\nabla}\nu)\,.\,\vec{H}_0 + (\vec{\nabla}\nu)\,.\,\vec{h}\Big)\Big\} + \nabla^2\vec{h}\,\Big] \\ + \Big[\,\nu\Big\{\Big(\vec{H}_0\,.\,\vec{\nabla}\Big)\,\frac{\partial\vec{u}}{\partial t} - \vec{H}_0\Big(\,\vec{\nabla}\,.\,\frac{\partial\vec{u}}{\partial t}\Big)\Big\} - \Big\{(\vec{\nabla}\nu)\,.\,\frac{\partial\vec{u}}{\partial t}\Big\}\,\vec{H}_0\,\Big]\end{aligned} \tag{12}$$

Assuming time dependence of $\vec{h}$ also to be given by

$$\vec{h} = \vec{\bar{h}}\,\exp\,(-i\omega t), \tag{13}$$

the field equations (5) and (12) can be put in the form

$$Lv = 0 = \begin{bmatrix} M & P \\ N & Q \end{bmatrix}\begin{bmatrix} \vec{\bar{u}} \\ \vec{\bar{h}} \end{bmatrix} \tag{14}$$

subject to the constraint

$$\vec{\nabla}\ \{\ 1/\nu\ (\vec{\nabla}\nu)\ .\ \vec{H}_0\ \} = 0 \tag{15}$$

where

$$M(\bar{\lambda},\ \bar{\mu},\ \rho) = (\bar{\lambda}+\bar{\mu})\vec{\nabla}\ (\vec{\nabla}\ .\ )+\bar{\mu}\nabla^2+(\vec{\nabla}\ \bar{\lambda})\ (\vec{\nabla}\ .\ )$$

$$+\vec{\nabla}\ \bar{\mu}\times(\vec{\nabla}\times)+2(\vec{\nabla}\ \bar{\mu}\ .\ \vec{\nabla})+\rho\omega^2$$

$$P(\nu) = -\nu\vec{H}_0\times)\vec{\nabla}\times)$$

$$N(\sigma,\ \nu) = i\omega\sigma\ [\ \nu\ \{\ (\vec{H}_6\ .\ \vec{\nabla})-\vec{H}_0\ (\vec{\nabla}\ .\ )\ \}-\vec{H}_0\ \{\ (\vec{\nabla}\ \nu)\ .\ \}\ ] \tag{16}$$

$$Q(\sigma,\ \nu) = -[\ \nabla^2+\vec{\nabla}\ \{\ 1/\nu\ (\vec{\nabla}\ \nu)\ .\ \}+i\omega\sigma\nu\ ]$$

Equation (15) clearly defines a necessary restriction for such time-harmonic fields to satisfy the equation.

Solution: We assume for the field equations (14)

$$(\bar{\lambda},\ \bar{\mu},\ \rho,\ \nu,\ \sigma,\ Y_s,\ Y_v) = (\bar{\lambda}_0,\ \bar{\mu}_0,\ \rho_0,\ \nu_0,\ \sigma_0,\ Y_s^0,\ Y_v^0)$$

$$+\epsilon\ (\bar{\lambda}_1,\ \bar{\mu}_1,\ \rho_1,\ \nu_1,\ \sigma_1,\ Y_s^{(1)},\ Y_v^{(1)}) \tag{17}$$

such that

$$0 = <\bar{\lambda}_1> = <\bar{\mu}_1> = <\rho_1> = <\nu_1> = <Y_s^{(1)}> = <Y_v^{(1)}>$$

where

$$<\sigma_1> = \sigma_1{}' \neq 0, \qquad \sigma_0 = 0. \tag{18}$$

From (18) it is clear that we have chosen to consider the case of weak electrical conductivity of the medium. (For $\sigma_0 \neq 0$, the effect of randomness arises in the second order terms; this is being studied separately).

Thus we obtain

$$L_0, L_1, L_2 \equiv \begin{bmatrix} M_0 & P_0 \\ 0 & Q_0 \end{bmatrix},\quad \begin{bmatrix} M_1 & P_1 \\ N_1 & Q_1 \end{bmatrix},\quad \begin{bmatrix} 0 & 0 \\ N_2 & Q_2 \end{bmatrix} \tag{19}$$

where

$$M_0 = (\bar{\lambda}_0+\bar{\mu}_0)\vec{\nabla}(\vec{\nabla}\ .\ )+\bar{\mu}\nabla^2+\rho_0\omega^2, \qquad M_1 = M(\bar{\lambda}_1,\ \bar{\mu}_1,\ \rho_1)$$

$$P_0 = P(\nu_0), \qquad P_1 = P(\nu_1)$$

$$N_1\ (\sigma_1,\ \nu_0) = i\omega\sigma_1\ \nu_0\ [\ (\vec{H}_0\ .\ \vec{\nabla})-\vec{H}_0\ (\vec{\nabla}\ .\ )\ ]$$

$$N_2 = N(\sigma_1,\ \nu_1), \qquad Q_0 = -\nabla^2$$

$$Q_1 = -\ [\ i\omega\sigma_1\ \nu_0+(1/\nu_0)\ \vec{\nabla}\ \{\ (\vec{\nabla}\ \nu_1)\ .\ \}\ ]$$

$$Q_2 = -i\omega\sigma_1\ \nu_1+(1/\nu_0^2)\ \vec{\nabla}\ \{\ \nu_1\ (\vec{\nabla}\nu_1)\ .\ \} \tag{20}$$

so that

$$<Lj> = \begin{bmatrix} 0 & 0 \\ <Nj> & <Qj> \end{bmatrix}, \quad j = 1, 2. \tag{21}$$

and

$$<N_1> = N_1(\sigma'_1, \nu_0 , \qquad <Q_1> = -i\omega\sigma'_1 \, \nu_0$$

$$<N_2> = i\omega \, [<\sigma_1 \, \nu_1> \{ (\vec{H}_0 \, . \, \vec{\nabla}) - \vec{H}_0 \, (\vec{\nabla} \, . \,) \} - \vec{H}_0 <\sigma_1 \{ (\vec{\nabla} \nu_1) \, . \, \}>]$$

$$<Q_2> = -i\omega <\sigma_1 \, \nu_1> + (1/\nu_0^2) \, \vec{\nabla} \, \{<\nu_1 \, (\vec{\nabla} \, \nu_1) > . \, \} \tag{22}$$

Assuming plane wave propagation we get

$$< v(\vec{x}) > = \begin{bmatrix} \vec{A} \\ \vec{T} \end{bmatrix} \exp . \, (i\vec{K} \, . \, \vec{x}) \tag{23}$$

Taking $f = 0$ in equation (3) and considering terms upto the order of ϵ only we obtain two equations involving $\vec{A}$ and $\vec{T}$:

$$(\bar{\lambda}_0 + \bar{\mu}_0) \, (\vec{K} \, . \, \vec{A}) \, \vec{K} + K^2 \, \bar{\mu}_0 \, \vec{A} - \rho_0 \, \omega^2 \, \vec{A} + i\nu_0 \, \vec{H}_0 \times (\vec{K} \times \vec{T}) = 0 \tag{24}$$

and

$$K^2 \, \vec{T} + \epsilon\omega\sigma_1' \, \nu_0 \, [\, (\vec{K} \, . \, \vec{H}_0) \, \vec{A} - \vec{H}_0 \, (\vec{K} \, . \, \vec{A}) + i \, \vec{T} \,] = 0 \tag{25}$$

Now eliminating $\vec{T}$ from (24) and (25) we obtain

$$[\, (\bar{\lambda}_0 + \bar{\mu}_0) \, (\vec{K} \, . \, \vec{A}) \, \vec{K} + (\bar{\mu}_0 \, K^2 - \rho_0 \, \omega^2) \, \vec{A} \,] \, K^2 + i\omega\sigma'_1 \, \nu_0^2 \, \epsilon \, [\, \{ \, (\vec{K} \, . \, \vec{A}) \, H_0^2$$

$$- (\vec{K} \, . \, \vec{H}_0) \, (\vec{A} \, . \, \vec{H}_0) \, \} \, \vec{K} + (\vec{K} \, . \, \vec{H}_0)^2 \, \vec{A} - (\vec{K} \, . \, \vec{H}_0) \, (\vec{K} \, . \, \vec{A}) \, \vec{H}_0 \,] = 0. \tag{26}$$

We choose

$$\vec{K} = \vec{K}_i, \; \vec{H}_0 = H_{01} \, \vec{i} + H_{02} \, \vec{j}, \; \vec{A} = A_1 \, \vec{i} + A_2 \, \vec{j} + A_3 \, \vec{k} \tag{27}$$

so that $\vec{i}$ is in the direction of propagation and $\vec{i}$, $\vec{j}$ define the plane of $\vec{H}_0$ and the direction of propagation. Equation (26) now splits itself into three equation

$$[\, (\bar{\lambda}_0 + 2\bar{\mu}_0) K^2 - \rho_0 \omega^2 \,] \, A_1 + i\omega\sigma_1' \, \nu_0^2 \, \epsilon \, H_{02} \, [\, A_1 \, H_{02} - A_2 \, H_{01} \,] = 0 \tag{28}$$

$$[\, \bar{\mu}_0 \, K^2 - \rho_0 \, \omega^2 \,] \, A_2 + i\omega\sigma_1' \, \nu_0^2 \, \epsilon \, H_{01} \, [\, A_2 \, H_{01} - A_1 \, H_{02} \,] = 0 \tag{29}$$

$$[\, \bar{\mu}_0 \, K^2 - \rho_0 \, \omega^2 \,] \, A_2 + i\omega\sigma_1' \, \nu_0^2 \, \epsilon \, H_{01}^2 \, A_3 = 0 \tag{30}$$

Eliminating A_1, A_2 from (28) and (29) we obtain

$$[\, (\bar{\lambda}_0 + 2\bar{\mu}_0) \, K^2 - \rho_0 \, \omega^2 \,] \, [\, \bar{\mu}_0 \, K^2 - \rho_0 \omega^2 \,]$$

$$+ \epsilon . \, i\omega\sigma_1' \, \nu_0^2 \, [\, H_{01}^2 \, \{ \, (\bar{\lambda}_0 + 2\bar{\mu}_0) \, K^2 - \rho_0 \omega^2 \, \} + H_{02}^2 \, (\bar{\mu}_0 K^2 - \rho_0 \omega^2) \,] = 0 \tag{31}$$

The solutions of this equation, upto first order in ϵ, are

$$K = K_c - i\delta_c$$

where

$$K_c = \surd[\rho_0/(\bar{\lambda}+2\bar{\mu}_0)]\,.\,\omega, \quad \delta_c = \epsilon\,.\,\{\sigma_1' v_0^2 H_{02}^2/2\surd[\rho_0(\bar{\lambda}_0+2\bar{\mu}_0)]\} \tag{32}$$

and

$$K_s = K_s - i\delta_s \tag{33}$$

where

$$K_s = \surd(\rho_0/\bar{\mu}_0)\,.\,\omega, \quad \delta_s = \epsilon\,.\,\{\sigma_1' v_0^2 H_{01}^2/2\surd(\rho_0\bar{\mu}_0)\}$$

(It is to be noted that K_c, δ_c, K_s, δ_s are all complex quantities).

Analysis of the dispersion equations (30) **and** (31) : 1. Equations (28) and (29) show that the motions along $\vec{i}$ and $\vec{j}$ directions are coupled and propagate two possible velocities. These velocities are obtainable from the frequency equation (31).

From (28) and (32) we obtain

$$\frac{A_1}{A_2} = \epsilon\,.\,\frac{i\sigma_1' v_0^2 H_{02}^3}{4\rho_0\omega H_{01}} = o(\epsilon) \tag{34}$$

Thus, (A_2/A_1) is small of the order ϵ and hence waves for which K is determined by (32) are mainly longitudinal since the longitudinal amplitude A_1 is much greater than the transverse amplitude A_2. Similarly from (29) and (33) we obtain

$$\frac{A_1}{A_3} = \epsilon\,.\,\frac{i\sigma_1' v_0^2 H_{01}^3}{4\rho_0\,\omega H_{O2}} = o(\epsilon). \tag{35}$$

Therefore (A_1/A_2) is small of the order ϵ and hence waves for which K is determined by (33) are mainly transverse.

2. If the magnetic field is longitudinal, i.e. if $H_{02} = 0$, then from (28) we observe that motion is undisturbed by the magnetic field in this direction and waves propagate with velocity determined by

$$(\bar{\lambda}_0+2\bar{\mu}_0)\,K^2 = \rho_0\,\omega^2 \tag{36}$$

If the magnetic field is transverse i.e., if $H_{01} = 0$ then from (29) we observe that motion is undisturbed by the magnetic field in this direction and waves propagate with velocity determined by

$$\bar{\mu}_0\,K^2 = \rho_0\omega^2 \tag{37}$$

3. *Motion along k-direction*: Motion takes place along k-direction only with the single velocity determined by equation (30).

4. *The case of elastic solid*: In the case of an elastic solid we take $\bar{\lambda}_0$ and $\bar{\mu}_0$ to be the usual Lame' constants. Equation (31) now reduces to the dispersion equation for waves propagating in a random magneto-elastic medium.

Putting

$$K = K_s - i\delta_s, \qquad K_s = \sqrt{(\rho_0/\bar{\mu}_0)} \cdot \omega$$

in (31) and retaining only first order terms, we obtain

$$\delta_s = \epsilon \cdot \frac{\sigma_1' \, \nu_0^2 \, H_{01}}{2\sqrt{(\rho_0 \, \bar{\mu}_0)}} \tag{38}$$

Since $\delta_s (>0)$, the amplitude of the wave increases exponentially as it advances unless $\sigma_1' = 0$. A similar result is true for

$$K = K_c - i\delta_c, \qquad K_c = \sqrt{(\rho_0/(\bar{\lambda}_0 + 2\bar{\mu}_0))} \cdot \omega.$$

(The case $\sigma_1' = 0$ is being examined separately). This result is different from that obtained in the visco-elastic case since in this case $\delta_{s,c}$ is the only term in the attenuation coefficient (See next section).

A particular case : Voigt solid : For the Voigt type of visco-elastic solid we have

$$S_{ij} = 2G\,(e_{ij} + \theta\, \ddot{e}_{ij}) \tag{39}$$

and hence

$$\bar{\lambda}_0 - \tfrac{1}{3}(Y_v^0 - Y_s^0) = \tfrac{1}{3}\left[\,\bar{G}^\circ - G^\circ - i\omega\,(\bar{G}^\circ\,\bar{\theta}^\circ - G^\circ\,\theta^\circ)\,\right]$$

$$\bar{\mu}_0 = \tfrac{1}{2}\, Y_s^\circ = G^\circ\,(1 - i\omega\theta^\circ) \tag{40}$$

where $G^\circ, \theta^\circ, \bar{G}^\circ, \bar{\theta}^\circ$ are the non-random parts of $G, \theta, \bar{G}, \bar{\theta}$ respectively. Thus

$$K_c - i\delta_c = K_c' - i\delta_c'$$

and

$$K_s - i\,\delta_s = K_s' - i\,\delta_s'$$

where $K_c', \delta_c', K_s' . \delta_s'$ are all real quantities given by

$$K_c' = \sqrt{(3\rho_0)}\, \frac{a\omega}{a^2+b^2} - \epsilon\sqrt{\left(\frac{3}{\rho_0}\right)} \cdot \frac{\sigma_1'\, \nu_0^2\, H_{02}^2}{2} \cdot \frac{b}{a^2+b^2}$$

$$\delta_c' = \sqrt{(3\rho_0)}\frac{b\omega}{a^2+b^2} + \epsilon\sqrt{\left(\frac{3}{\rho_0}\right)} \cdot \frac{\sigma_1'\, \nu_0^2\, H_{02}^2}{2} \cdot \frac{a}{a^2+b^2}$$

$$a = \frac{1}{\sqrt{2}}\left[(\bar{G}^\circ + 5G^\circ) + \sqrt{\{(\bar{G}^\circ + 5G^\circ)^2 + \omega^2(\bar{G}^\circ\,\bar{\theta}^\circ + 5G^\circ\,\theta^\circ)^2\}}\,\right]^{1/2}$$

$$b = \frac{-1}{\sqrt{2}}\left[-(\bar{G}^\circ + 5G^\circ) + \sqrt{\{(\bar{G}^\circ + 5G^\circ)^2 + \omega^2(\bar{G}^\circ\,\bar{\theta}^\circ + 5G^\circ\,\theta^\circ)^2\}}\,\right]^{1/2} \tag{42}$$

$$K_s' = \sqrt{\left(\frac{\rho_0}{G_0}\right)} \cdot \frac{a'\omega}{a'^2+b'^2} - \epsilon\frac{\sigma_1'\, \nu_0^2\, H_{01}^2}{2\sqrt{(\rho_0 G_0)}} \cdot \frac{b'}{a'^2+b'^2}$$

$$\delta_s' = \sqrt{\left(\frac{\rho_0}{G_0}\right)} \cdot \frac{b'\omega}{a'^2+b'^2} + \epsilon\frac{\sigma_1'\, \nu_0^2\, H_{01}^2}{2\sqrt{(\rho_0 G^0)}} \cdot \frac{a'}{a'^2+b'^2} \tag{43}$$

$$a' = \frac{1}{\sqrt{2}}\left(1 + \sqrt{(1+\omega^2\,\theta^{\circ 2})}\right)^{1/2}, \qquad b' = -\frac{1}{\sqrt{2}}\left(\sqrt{(1+\omega^2\theta^{\circ 2})} - 1\right)^{1/2}$$

Thus, waves for which K is determined by (32) in the Voigt type of visco-elastic medium admit of attenuation, the attenuation coefficient δ_c' being given by (42). Clearly, $\delta_c' < 0$ since $b < 0$; the first term in δ_c' is predominant. We conclude that decrease in attenuation coefficient is proportional to σ_1' and $(\nu_0 H_{02})^2$. Similarly waves for which K is determined by (33) in the Voigt type of material admit of attenuation, the decrease in attenuation coefficient in this case being proportional to σ_1' and $(\nu_0 H_{01})^2$.

It can be shown that waves for which K is determined by (32) or (33) in the Maxwell type of visco-elastic medium admit of attenuation and the decrease in attenuation coefficient is proportional to $\sigma_1' \nu_0^2 H_{02}^2$ or $\sigma_1' \nu_0^2 H_{01}^2$ in the two cases respectively.

Moreover, the phase velocity is altered by the action of magnetic field in both the Voigt and the Maxwell type of visco-elastic media. The decrease in phase velocity due to randomness is proportional to $\sigma_1' \nu_0^2 H_{02}^2$ for waves for which $K = K_c' - i\delta_c'$ and to $\sigma_1' \nu_0^2 H_{01}^2$ for waves for which $K = K_s' - i\delta_s'$ for both Voigt and Maxwell type of solids.

The attenuation of waves for which K is determined by (30) and (33) (for Voigt type of solids) is the same as obtained in (43).

Acknowledgement : The author expresses his gratitude to Dr. M. Mitra, Professor of Mathematics, Presidency College, Calcutta, for his guidance in the preparation of this paper.

References

Bhattacharyya, R, K. (1979) : *Journal of Technology* 24, 2.

Chow, P. L. (1973) : *Int. J. Engg. Sci.* (II), 953.

Karal, F. C. & Keller, J. B. (1964) : *J. Math. Phys.* (5) 4, 537,

Knopoff, L. (1955) : *Geophy, Res.* (60), 441.

Lessen, M. (1957) : Proc. Third Midwestern Conf. on Solid Mechanics (20).

Paria, G. (1962) : *Proc. Camb. Phil. Soc.* (58), 527.

Sinha, D. K. (1964) : *Pure. and Applied Geophysics* (57), 1, 53.

Willson, A. J. (1963) : *Proc. Camb. Phil. Soc.* (59), 483.

Van Kampen, N. G. (1976) : *Physics Reports* (Sec. C of *Phys. Letters*) (3) 24, 171.

DEPARTMENT OF MATHEMATICS,
BRAHMANANDA KESHAB CHANDRA COLLEGE,
CALCUTTA-700 035.

Bull. Cal. Math. Soc., 75, 257–258 (1983)

EXTENSION OF AN IDENTITY OF REDEI FOR LAGUERRE POLYNOMIAL

TULSHIDAS BANERJEE

(*Received 19 November 1979*)

1. Introduction : In a recent paper L. B. Redei has shown the representation for simple Laguerre Polynomials

$$L_n(x) = (-)^n \frac{e^x}{n!}\left[x\,\frac{d^2}{dx^2}+\frac{d}{dx}\right]^n e^{-x} \tag{1}$$

where $L_n(x)$ may be defined by the Redrigue's formula

$$L_n(x) = \frac{e^x}{n!}\left(\frac{d}{dx}\right)^n (x^n e^{-x}) \tag{2}$$

The object of this paper is to show that Redei's representation (1) for Simple Laguerre Polynomials is a particular case of our extended result which is stated below :

We proceed to show the representation involving General Laguerre Polynomials :

$$L_n^{(\alpha)}(x) = (-1^n \frac{e^x x^{-\alpha}}{n!}\left[\,x\frac{d^2}{dx^2}+(1-\alpha\ \ \frac{d}{dx}\right]^n e^{-x}x^{\alpha} \tag{3}$$

where $L_n^{(\alpha)}(x)$ may be defined dy

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\,D^n\,(x^{\alpha+n}\,e^{-x}) \tag{4}$$

Obviously (3) is an extension of (1), because (3) reduces to (1) when $\alpha=0$.

2. Proof of the Result (3) : Using (4) we write equation (3) in the form

$$A_n(x) = (-1)^n\,D^n\,(x^{\alpha+n}\,e^{-x}) \tag{5}$$

where $A_n(x)$ is defined by

$$A_n(x)=\left[\,x\frac{d^2}{dx^2}+(1-\alpha)\,\frac{d}{dx}\right]^n x^{\alpha}\,e^{-x}. \tag{6}$$

Thus it is sufficient to prove (5) by method of induction.

Equation (5) is true for $n=0$ and it is found to be true for $n=1$. Next if we assume that the result is true for n, then we are to show that the result is also true for $(n+1)$.

By definition it follows that

$$A_{n+1}(x)=\left[x\frac{d^2}{dx^2} +(1-\alpha)\frac{d}{dx}\right](-1)^n D^n (x^{n+\alpha} e^{-x})$$

$$=(-1)^n\left[x\frac{d^2}{dx^2}+(1-\alpha)\frac{d}{dx}\right] D^n (x^{n+\alpha} e^{-x})$$

$$=(-1)^n\left[x\frac{d^{n+2}}{dx^{n+2}}(x^{n+\alpha} e^{-x})+(1-\alpha)\frac{d^{n+1}}{dx^{n+1}}(x^{n+\alpha} e^{-x}\right]$$

Now for the first member in the right hand side of the above equation we use the identity

$$D^n f(xf(x)) = n\frac{d^{n-1}}{dx^{n-1}} f(x)+x\frac{d^n}{dx^n} f(x),$$

which is valid for any smooth function $f(x)$ and obtain

$$A_{n+1}(x) = (-1)^n\left[\frac{d^{n+2}}{dx^{n+2}}(x^{n+\alpha+1} e^{-x})-(n+\alpha+1)\left(\frac{d}{dx}\right)^{n+1}(x^{n+\alpha}e^{-x})\right]$$

Since

$$\frac{d}{dx}(x^{n+\alpha+1} e^{-x}) = (n+\alpha+1)x^{n+\alpha} e^{-x}-x^{\alpha+n+1} e^{-n},$$

it follows that

$$A_{n+1}(x) = (-1)^{n+1} D^{n+1}(x^{n+\alpha+1} e^{-x}),$$

which proves our assertion.

Added in Proof : After submitting the paper it came to our notice that a result equivalent to our result was given by O. V. Viskov (Acta Sci Math. (Szeged)) 39 (1977), 27-28, by a different method.

Acknowledgement : I am grateful to Dr. S. K. Chatterjea of the Department of Pure Mathematics of the Calcutta University, in the preparation of this paper for his valuable suggestions.

Reference :

Redei, L. B.: An Identity for Laguerre Polynomial—Acto Scientiarum Mathamaticarum.

DEPARTMENT OF PHYSICS,
PRESIDENCY COLLEGE,
CALCUTTA-700073.

Bull. Cal. Math. Soc., **75** 259–263, (1983)

A FEW REMARKS ON THE CONTINUITY CONDITION FOR A COMPRESSIBLE FLUID IN A STATE OF TURBULENCE

A. Mokid

(*Received 17 September 1979*)

Abstract : The statistical concept pertaining to turbulent fluid flow having homogeneous and isotropic features gives rise to a relation of the form :

$$g = f + \frac{r}{2}\frac{\partial f}{\partial r}$$

where

$$\overline{u^2}\, f(r, t) = \overline{u_1 u_1'}, \quad \overline{u^2}\, g(r, t) = \overline{u_2 u_2'} = \overline{u_3 u_3'},$$

$\overline{u^2}$ being the mean square velocity.

In the current work, we have extended this relation for the case of compressibility with an added term which will incidentally count for the measure of the compressibility for the fluid under consideration.

1. Introduction : Osborne Reynolds (1894) was the first who conceived turbulent motion as a simple superposition of fluctuating motions over mean motion. Subsequently, L. Prandtl (1925) developed momentum transfer theory to picture turbulent characteristics in comparison to the kinetic theory of gases. Extending this concept, G. I. Taylor (1932) formulated a two-dimensional theory of turbulence in which vortices instead of momentum were assumed to be identifiable and in that sense it would not be lost to the neighbourhood for a certain specific average distance in course of motion of turbulent eddies. This was incidentally termed as vorticity transfer theory and this gave somewhat better results than that obtained through momentum transfer theory. This theory also failed to solve the intricacies of the problem of turblence. Accordingly, both these transfer theories gave satisfactory solutions in case of a few number of engineering and meteorological problems only. However a deeper analysis of the phenomena of turbulence was still lacking. For this reason, G. I. Taylor (1935) proposed a statistical apprach to the problem of turbulence, and T. V. Karman and L. Howarth (1938) extended the same.

In the aforesaid works, G. I. Taylor (1935), T. V. Karman and L. Howarth (1938) calculated the double velocity correlations for a homogeneous and isotropic turbulent flow relating to an incompressible fluid in regard to two points and the same instant of time. H. P. Robertson (1940) also calculated the same by applying the theory of invariants for all possible translations and rotations (both for proper and improper rotations) of the rigid frame under reference. Extending these concepts, a relation between the longitudinal and the lateral velocity correlations was given theoretically and this was subsequently verified through measurements at the farthest and from the entrance face of the wind-tunnel (beyond 20 mesh lengths or so).

In the present paper, an attempt has been made to obtain a relation between the longitudinal and the lateral velocity correlations for a homogeneous and isotropic turbulence in case, the fluid is considered to be compressible one. The velocity correlations in this context are calculated for two neighbouring points at the same instant of time. Further, the on-side and cross-correlations are deduced and compared theoretically.

2. Formulaiton of the problem : The equation of continuity at a point $P(\vec{X}, t)$ in case of a compressible fluid is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i} (\rho u_i) = 0 \tag{1}$$

$$\text{or} \quad \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0$$

$$\text{or} \quad \rho \frac{\partial u_i}{\partial x_i} = -\left(\frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i} \right)$$

$$\text{or} \quad \frac{\partial u_i}{\partial x_i} = -\frac{1}{\rho} \frac{D\rho}{Dt}$$

$$= -\frac{D}{Dt}\, l_n \rho$$

$$= F_1 (\vec{X}, t), \text{ (say)}, \tag{2}$$

where $(\vec{X} = x_i,\ i = 1, 2, 3)$ and u_i s the i-th component of fluctuating velocity at the point P having co-ordinates $x_i\ (= \vec{X})$ at time t, ρ being the density at the point P at the instant t,, and

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u_i \frac{\partial \rho}{\partial x_i}$$

Similarly, for the second point $P'(\vec{X}', t)$, we obtain

$$\frac{\partial u'_j}{\partial x'_j} = -\frac{1}{\rho'} \frac{D\rho'}{Dt}$$

$$= -\frac{D}{Dt}\, l_n \rho'$$

$$= F_2 (\vec{X}', t) \tag{3}$$

where $(\bar{X}' = x_j', j = 1, 2, 3)$ and u_j' is the j-th component of velocity at the point

P' having coordinates $x_j' \,(=\vec{X}')$ at time t, and ρ' is the density at the point P' at the same time t. It is to be noted in this context that $\frac{D\rho'}{Dt}$ is explicitly given by

$$\frac{D\rho'}{Dt} = \frac{\partial \rho'}{\partial t} + u_j' \frac{\partial \rho'}{\partial x_j'}$$

Multiplying (2) by (3), we obtain after averaging,

$$\overline{\frac{\partial u_i}{\partial x_i} \frac{\partial u'_j}{\partial x'_j}} = \overline{F_1 (\vec{X}, t)\, F_2 (\vec{X}', t)}$$

$$\text{or,} \quad \frac{\partial^2 \overline{u_i u'_j}}{\partial x_i \partial x'_j} = \overline{F_1 (\vec{X}, t)\, F_2 (\vec{X}', t)} \tag{4}$$

Since the turbulence is assumed homogeneous and isotropic, we put

$$-\frac{\partial}{\partial x_i} = \frac{\partial}{\partial x'_i} = \frac{\partial}{\partial \xi_i} \quad \text{and} \quad x'_i - x_i = \xi_i = |\vec{r}\,|$$

in equation (4), and obtain

$$-\frac{\partial^2 \overline{u_i u'_j}}{\partial \xi_i \partial \xi_j} = \bar{F}(r, t) \tag{5}$$

Now, $\overline{u_i u_j'} = \overline{u^2} R_{ij}$, $\overline{u^2}$ meaning the mean square velocity.

$$\therefore \frac{\partial^2 R_{ij}}{\partial \xi_i \partial \xi_j} = -\frac{1}{\overline{u^2}} \bar{F}(r, t)$$

$$\text{or.} \quad \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left[\frac{f-g}{r^2}\, \xi_i \xi_j + g\delta_{ij} \right] = -\frac{1}{\overline{u^2}} \bar{F}(r, t) \tag{6}$$

where $\overline{u^2} f(r, t) = \overline{u_1 u_1'}$ and $\overline{u^2} g(r, t) = \overline{u_2 u_2'} = \overline{u_3 u_3'}$.

Equation (6), on simplification, leads to

$$\left[\frac{1}{r} \frac{\partial^2 f}{\partial r^2} \frac{\xi_i}{r} - \frac{1}{r^2} \frac{\partial f}{\partial r} \frac{\xi_i}{r} + \frac{2}{r^2} \left(\frac{\partial f}{\partial r} - \frac{\partial g}{\partial r} \right) \frac{\xi_i}{r} - \frac{4}{r^3} (f-g) \frac{\xi_i}{r} \right] \xi_i + \left[\frac{1}{r} \frac{\partial f}{\partial r} + \frac{2}{r^2} (f-g) \right] \delta_{ii} = -\frac{1}{\overline{u^2}} \bar{F}(r, t) \tag{7}$$

Without loss of generality, as usual we may take the line joining P and P' as 1-direction and 2, 3 directions as two mutually perpendicular directions orthogonal to 1-direction. Hence, without loss of generality, let us set forth

$|\vec{r}|=\xi_1$ and $\xi_2=\xi_3=0$. Thus, putting $i=1$ in (6), we obtain

$$\frac{\partial^2 f}{\partial r^2}-\frac{1}{r}\,\frac{\partial f}{\partial r}+\frac{2}{r}\left(\frac{\partial f}{\partial r}-\frac{\partial g}{\partial r}\right)-\frac{4}{r^2}\,(f-g)$$

$$+\frac{1}{r}\,\frac{\partial f}{\partial r}+\frac{2}{r^2}\,(f-g)$$

$$=-\frac{1}{u^2}\,\bar{F}(r,t)$$

$$\text{or,}\quad \frac{\partial^2 f}{\partial r^2}+\frac{2}{r}\left(\frac{\partial f}{\partial r}-\frac{\partial g}{\partial r}\right)-\frac{2}{r^2}\,(f-g)$$

$$=-\frac{1}{u^2}\,\bar{F}(r,t) \tag{8}$$

$$\text{or,}\quad \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right)+2\,\frac{\partial}{\partial r}\left[\frac{1}{r}\,(f-g)\right]$$

$$=-\frac{1}{u^2}\,\bar{F}(r,t)$$

Multiplying (8) throughout by δr and integrating between the limits 0 to r, we obtain

$$\frac{\partial f}{\partial r}+\frac{2}{r}\;(f-g)=-\frac{1}{u^2}\int_0^r \bar{F}(r,t)\,dr \tag{9}$$

It is well-krown that

$$f=1+\frac{1}{2\,!}\,f_0^{\mathrm{II}}\,r^2+\frac{1}{4\,!}\,f_0^{\mathrm{IV}}r^4+\cdots$$

$$g=1+\frac{1}{2\,!}\,g_0^{\mathrm{II}}\,r^2+\frac{1}{4\,!}\,g_0^{\mathrm{IV}}r^4+\cdots$$

where $\frac{\partial f}{\partial r}$ and $\frac{2}{r}\;(f-g)$ are as usual zero at $r=0$.

From the relation (9), we can draw the following conclusions :

(i) From wind tunnel measurements, one can determine g and f and hence $\bar{F}(r,t)$ can be calculated with the help of l.h. expression of (9) after differentiation of $f(r,t)$ with respect to r and making necessary insertion. Accordingly, the value so obtained for $\bar{F}(r,t)$ is a messure of compressibility of the fluid under consideration.

(ii) When the fluid is incomprressible, it is obvious that

$$\frac{D\rho}{Dt}=\frac{D\rho'}{Dt}=0$$

and hence,

$$\bar{F}(r,t)=0$$

In case of such simpilfied picture of an incompressible fluid, we need verify

$$\frac{\partial f}{\partial r}+\frac{2}{r}\,(f-g)=0$$

i.e.

$$g=f+\frac{r}{2}\,\frac{\partial f}{\partial r} \tag{10}$$

which is, in fact, a relation previously obtained by T. V. Karman and L. Howarth (1938) as well as H. P. Robertson (1940) for an incompressible fluid in a state of turbulence which is homogeneous and isotropic in nature. This relation (10) will then testify whether the fluid under consideration can be treated as incompressible one.

Acknowledgement : The author is greatly indebted to Dr. K. M. Ghosh of Calcutta for suggesting the problem as well as for guidance during the course of this paper.

References

Karman, T. V. and Howarth, L. (1938) : *Proc. Roy. Soc.* **A 164**, 192.

Prandtl, L. (1925) : *ZAMM*. Ed. 5, 136.

Reynolds, O. (1894) : *Phil. Trans.* AC IXXXVI, 123.

Robertson, H. P, (1940) : *Proc. Camb. Phil. Soc.* 36, 209.

Taylor, G . I. (1932) : *Proc. Roy. Soc.* A 135, 655.

— — (1935) : *Proc. Roy. Soc.* A 151, 421.

DEPARTMENT OF MATHEMATICS
RAMPURHAT COLLEGE
P. O. RAMPURHAT
DIST. BIRBHUM

Bull. Cal. Math. Soc.. **75**, **264—270** (**1983**)

UNSTEADY MAGNETO-HYDRODYNAMIC FLOW THROUGH POROUS MEDIA IN A CHANNEL WHOSE CROSS-SECTION IS A CIRCULAR SECTOR

PREM CHAND GUPTA

(*Received 3 December 1979 ; Revision Received 28 April 1980*)

Abstract : In this paper, unsteady laminar flow of a viscous incompressible fluid through porous media having uniform radial magnetic field in a channel whose cross-section is a circular sector $(-\alpha \leqslant \theta \leqslant \alpha,\ 0 \leqslant r \leqslant a)$ under the influence of arbitrary time varying pressure gradient has been studied with the help of the technique of Laplace, Finite Fourier cosine and the Finite Hankel transforms. When the pressure gradient is an absolute constant, it has been observed that for a fixed value of Hartmann number, the flow increases as porosity increases and that for a fixed porosity, the flow decreases as Hartmann number increases.

1. Introduction :

The study of flow through porous media is of considerable interest in the fields of petroleum engineering concerned with the movement of oil and gas, ground water hydrology, heat transfer in cooling systems and chemical engineering for filtration process etc It is due to this reason that several investigations are being done in this field.

Recently, Ahmadi & Manvi (1971) derived a general equation of motion for flow through porous medium and applied them to some basic flow problems. On account of varied practical applications of the magneto-hydrodynamic flow problems in pipes, Gulab Ram & Mishra, R. S. (1977) discussed the unsteady magneto-hydrodynamic flow through porous media in a circular channel with constant pressure gradient. In the present paper, the same flow has been discussed in a channel whose cross-section is a circular sector, the pressure gradient being an arbitrary function of time.

2. Governing equations of motion :

Using a cylindrical polar coordinates system (r, θ, z), the cross-section of the channel is formed by $r=a$ and $\theta=\pm\alpha$ (Fig.1). Taking the axis of the channel along the z-axis, the equation of motion for the laminar, incompressible flow in porous media under the influence of uniform radial magnetic field takes the form

$$\frac{\partial u_z}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}\right)-\frac{\nu}{K}u_z-\frac{\sigma}{\rho}\mu_e^2 H_0^2 u_z \tag{2.1}$$

where u_z is the fluid velocity along z-axis, p is the fluid pressure independent of r and θ, μ, ν and ρ are viscosity, kinematic viscosity and the density of the

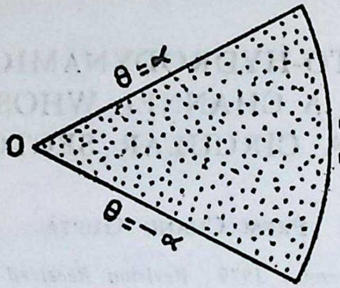


FIG.1 SHOWING THE CROSS-SECTION OF THE CHANNEL

fluid respectively, K is the permeability of the porous medium, μ_e is the magnetic permeability, H_0 is the uniform radial magnetic field and σ is the conductivity. The induced magnetic field has been ignored by taking σ to be small compared with the applied magnetic field. (Walker J. S. 1978)

Using the non-dimensional quantities

$$U=\frac{au_z}{\nu}\ ,\ z=\frac{z}{a}\ ,\ R=\frac{r}{a}\ ,\ p'=\frac{pa^2}{\rho\nu^2}\ ,\ \tau=\frac{t\nu}{a^2}$$

Equation (2.1) reduces to its non-dimensional form

$$\frac{\partial U}{\partial \tau}=f(\tau)+\left(\frac{\partial^2 U}{\partial R^2}+\frac{1}{R}\ \frac{\partial U}{\partial R}+\frac{1}{R^2}\ \frac{\partial^2 U}{\partial \theta^2}\right)-\left(\frac{1}{K_1}+m^2\right)U \tag{2.2}$$

where

$$K_1=\frac{K}{a^2}\ ,\ m=a\mu_e H_0\left(\frac{\sigma}{\mu}\right)^{1/2}=\text{Hartmann number}$$

and $-\dfrac{\partial p'}{\partial z}=f(\tau)=$ arbitrary function of time.

The initial and boundary conditions are :

(i) $\tau \leqslant 0,\ U(R,\ \theta,\ \tau)=0$ for $-\alpha \leqslant \theta \leqslant \alpha, 0 \leqslant R \leqslant 1$ (2.3)

(ii) $\tau > 0,\ U(R, \pm\alpha, \tau)=0$ for $0 \leqslant R \leqslant 1$ (2.4)

$U(0, \theta, \tau)=0$ for $-\alpha \leqslant \theta \leqslant \alpha$ (2.5)

$U(1, \theta, \tau)=0$ for $-\alpha \leqslant \theta \leqslant \alpha$ (2.6)

Due to the symmetry about the plane $\theta=0$, the flow in the region $0 \leqslant \theta \leqslant \alpha$ is considered and the boundary condition (2.4) is equivalent to

$$U(R, \alpha, \tau)=0 \text{ for } 0 \leqslant R \leqslant 1 \tag{2.7}$$

and

$$\frac{\partial U}{\partial \theta}=0 \text{ for } \theta=0$$

2. Solution of the problem : Equation (2.2) is to be solved under the conditions (2.3), (2.5), (2.6) and (2.7). For this, we shall make use of the following transforms :

(i) Finite Fourier Cosine transform with respect to θ

$$\bar{U}(R, n, \tau)=\int_0^{\alpha} U(R, \theta, \tau) \cos q_n\theta d\theta \tag{3.1}$$

where

$$q_n=\frac{(2n+1)}{2\alpha}$$

with

$$\frac{\partial U}{\partial \theta}=0 \text{ at } \theta=0, \; U(R, \theta, 0)=0,$$

and

$$U(R, \alpha, \tau)=0 \text{ for } 0 \leqslant R \leqslant 1, \; \tau>0$$

(ii) Finite Hankel transform with respect to R

$$\bar{U}_H(p_i, n, \tau)=\int_0^1 \bar{U}(R, n, \tau) R J_{q_n}(Rp_i) dR \tag{3.2}$$

where J_{q_n} is the Bessel's function of the first kind and p_i is a positive root of $J_{q_n}(p_i)=0$ with

$$\bar{U}(R, n, 0)=0 \; ; \; \bar{U}(0, n, \tau)=0, \; \bar{U}(1, n, \tau)=0 \text{ for } \tau>0$$

(iii) Laplace transform with respect to τ

$$\bar{\bar{U}}_H(s)=\int_0^{\infty} \bar{U}_H(p_i, n, \tau) e^{-s\tau} d\tau, \; s>0 \tag{3.3}$$

with

$$\bar{U}_H(p_i, n, 0)=0$$

Taking the Finite Fourier Cosine transform (3.1) of equation (2.2), and then the Finite Hankel transform (3.2), we obtain

$$\frac{\partial \bar{U}_H}{\partial \tau}=\frac{(-1)^n}{q_n} f(\tau) \int_0^1 R J_{q_n}(Rp_i) dR-\left(p_i^2+\frac{1}{K_1}+m^2\right) \bar{U}_H \tag{3.4}$$

Now using a relation given by Erdelyi 1953, p. 90, we deduce

$$\int_0^1 R J_{q_n}(Rp_i) dR=\frac{1}{p_i} J_{q_n+1}(p_i) S_{1,q_n}(p_i) \tag{3.5}$$

where $S_{1,q_n}(p_i)$ is the Lommel function.

Now taking the Laplace transform of equation (3.4) and using the relation (3.5), we finally obtain

$$\bar{\bar{U}}_H(s)=\frac{(-1)^n}{q_n}\,\frac{J_{q_n+1}(p_i)\,S_{1,q_n}(p_i)}{p_i}\,\frac{\bar{f}(s)}{s+(p_i^2+(1/K_1)+m^2)} \tag{3.6}$$

where $\bar{f}(s)$ is the Laplace transform of $f(\tau)$.

Inverting the Laplace transform by Convolution theorem, we have

$$\bar{U}_H(p_i, n, \tau)=\frac{(-1)^n}{q_n}\,\frac{J_{q_n+1}(p_i)\,S_{1,q_n}(p_i)}{p_i}\int_0^{\tau} e^{-(p_i^2+(1/K_1)+m^2)\lambda} f(\tau-\lambda)\,d\lambda \tag{3.7}$$

Inverting the Hankel transform (Tranter 1951, p. 88), we have

$$\bar{U}(R, n, \tau)=\frac{(-1)^n}{q_n}\,2\sum_i \frac{J_{q_n}(Rp_i)\,S_{1,q_n}(p_i)}{p_i J_{q_n+1}(p_i)}\int_0^{\tau} e^{-(p_i^2+(1/K_1)m^2)\lambda} f(\tau-\lambda)\,d\lambda \tag{3.8}$$

Inverting the Finite Fourier Cosine transform (Sneddon 1974, p. 426), we obtain,

$$U(R, \theta, \tau)=\frac{4}{\alpha}\sum_{n=0}^{\infty}\sum_i \frac{(-1)^n}{q_n}\,\frac{J_{q_n}(Rp_i)\,S_{1,q_n}(p_i)\cos q_n\theta}{p_i\,J_{q_n+1}(p_i)}\times$$

$$\int_0^{\tau} e^{-(p_1^2+(1/K_1)+m^2)\lambda} f\tau-\lambda\}\,d\lambda \tag{3.9}$$

where summation over i is to be taken for the positive roots of

$$J_{q_n}(p_i)=0$$

4. Particular Case : Constant Pressure Gradient : Here $f(\tau)=C$

Substituting this value of $f(\tau)$ in equation (3.9), we get

$$U(R, \theta, \tau)=\frac{4C}{\alpha}\sum_{n=0}^{\infty}\sum_i \frac{(-1)^n}{q_n}\,\frac{J_{q_n}(Rp_i)\,S_{1,q_n}(p_i)\cos q_n\theta}{p_i\,J_{q_n+1}(p_i)}\times$$

$$\left\{\frac{1-e^{-(p_i^2+(1/K_1)+m^2)\tau}}{p_i^2+(1/K_1)+m^2}\right\} \tag{4.1}$$

Case (i) : $\alpha=\pi$

If we put $\alpha=\pi$, we obtain the magneto-hydrodynamic flow in porous medium through a circular channel having a rigid boundary wall given by the plane

$$\theta=\pm\,\pi \text{ (Fig. 2.)}$$

The velocity distribution is immediately obtained by putting $\alpha=\pi$ in equation (4.1).

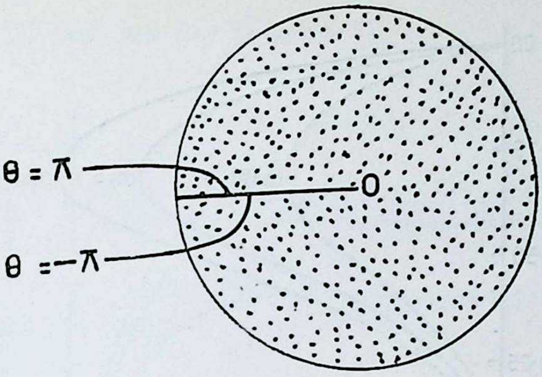


FIG 2

Case (ii) : $m \rightarrow 0$

If we put $m=0$ in equation (4.1), we obtain the flow in porous medium having no magnetic field.

Case (iii) : $K_1 \rightarrow \infty$

Resistancy of the porous medium is inversely proportional to K_1. If K_1 increases, the resistancy of the medium decreases. Thus if we put $K_1 \rightarrow \infty$ in equation (4.1), we obtain the magneto-hydrodynamic flow in ordinary medium (porosity 1) through a channel of cross-section as a sector of a circle.

Case (iv) : $K_1 \rightarrow \infty, m \rightarrow 0$

In this case the resistancy of the porous medium and the effect of magnetic field is zero and hence if we put $K_1 \rightarrow \infty, m=0$ in equation (4.1) we immediately obtain the ordinary flow through the channel of cross-section as a circular sector.

Discussion : Equation (3.9) gives the unsteady, laminar flow through porous medium having uniform radial magnetic field in a channel with cross-section as a circular sector under the influence of arbitrary time varying pressure gradient. Various flows can be obtained from this under different values of the pressure gradient $f(\tau)$. Equation (4.1) gives the same flow under the constant pressure gradient. Flow in a circular channel having a wall at the plane $\theta=\pi$ has been obtained in case (i). The flows in the absence of magnetic field, in the absence of resistancy of porous medium, and in the absence of resistancy of porous medium and magnetic field both, have been obtained in case (ii), (iii) and (iv) respectively.

Figures 3 and 4 give the idea about the nature of the velocity profiles and Fig. 5 about the unsteadiness of the motion given by equation (4.1). It is observed that, under constant pressure gradient, (i) at any instant, for a fixed porosity, the velocity decreases as Hartmann number increases (Figure 3), (ii) at

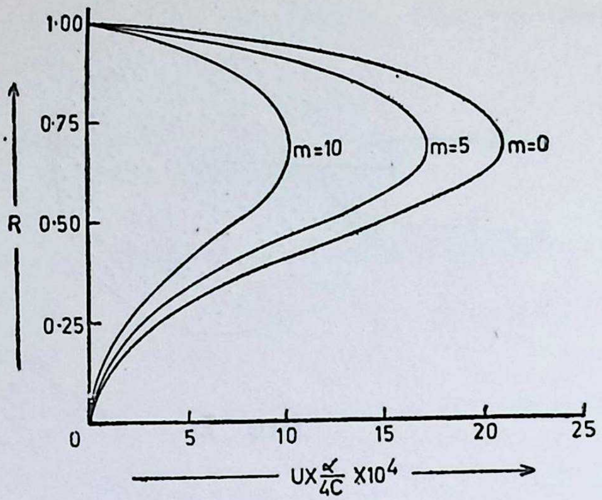


FIG.3 VELOCITY PROFILES PLOTTED AGAINST R AT τ=·02 FOR VALUES OF m WHEN $\alpha=\pi/6$, $\theta=\pi/18$, $K_1=1$ AND $f(\tau)=C$

any instant, for a fixed value of Hartmann number, the velocity increases as porosity increases (Figure 4), and (iii) unsteadiness of the motion is solely due

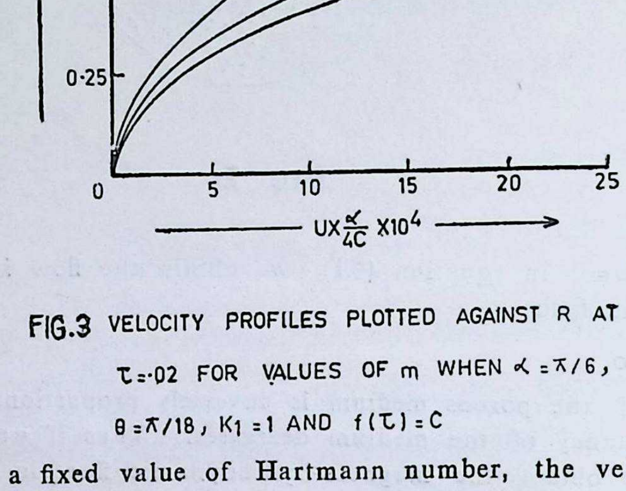


FIG. 4 VELOCITY PROFILES PLOTTED AGAINST R AT τ=·02 FOR VALUES OF K_1 WHEN $\alpha=\pi/6$, $\theta=\pi/18$, m=1 AND $f(\tau)=C$

to the impulsive start of the motion (Figure 5).

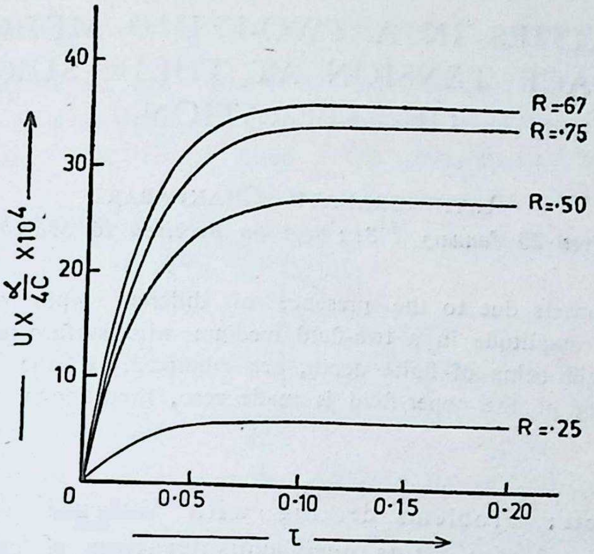


FIG. 5 VELOCITY PLOTTED AGAINST τ FOR VALUES OF R WHEN $\alpha = \pi/6$, $\theta = \pi/18$, $m = 1$, $K_1 = 1$ AND $f(\tau) = C$

Acknowledgement : Author feels pleasure to thank Dr Ram Gopal Sharma, Agra College, Agra, for his guidance and encouragement. Thanks are also due to the referee for his valuable suggestions in the improvement of this paper.

References

Ahmadi, G. & Manvi, R. (1971) : Equation of motion for viscous flow through a rigid porous medium, *Indian Journal of Technology*, 9, 441.

Erdelyi, A. *et al.* (1953) : Higher transcendental functions, II, McGraw Hill Book Co, New-York p. 90.

Gulab Ram & Mishra, R. S. (1977) : Unsteady flow through magneto-hydrodynamic porous media. *Indian Jour. Pure & Appl. Maths* 8, 6, 637.

Sneddon, I. N. (1974) : The use of Integral Transform, Tata McGraw Hill Pub. Co. Ltd. New Delhi.

Tranter, C. J. (1951) : Integral Transforms in Mathematical Physics, Mathuen & Co. Ltd. London.

Walker, J. S. (1978) : Solitary Fluid Transients in rectangular ducts with transverse magnetic fields, *ZAMP*, 29, 37.

DEPARTMENT OF MATHEMATICS
AGRA COLLEGE
AGRA, INDIA

Bull. Cal. Math. Soc., **75**, 271—282 (1983)

SINGULARITIES IN A TWO-FLUID MEDIUM WITH SURFACE TENSION AT THEIR SURFACE OF SEPERATION

RATHINDRANATH CHAKRABARTI

(*Received 20 January 1981 ; Revision Received 16 March 1981*)

Abstract : Velocity potentials due to the presence of different types of singularities oscillating harmonically with small amplitude in a two-fluid medium with surface tension at their surface of separation, the lower fluid being of finite depth, are obtained. If the surface tension term is neglected, and the density of the upper fluid is made zero, then known results obtained by other workers are recovered.

1. Introduction : Problems dealing with internal waves arising at the surface of separation of two fluids need consideration of different types of singularities in one of the two fluids. In case of bodies within the liquid, waves may be generated by the movement of the body or reflected from the body. The two cases are equivalent and the resulting motion can be described by a series of singularities placed within the body. The boundary conditions on the surface of separation (SS) give equations from which the exact form the the potential can be obtained.

Here the velocity potentials describing the singularities are obtained considering the effects of surface tension at the SS. Line and point singularities are considered when the lower fluid is of finite depth, the upper being of infinite height. The one-fluid problem was first considered by Throne (1953) and the corresponding problem where the effect of surface tension was considered by Rhodes-Robinson (1970). The two-fluid problem was considered by Gorgu and Kassem (1978). The present problem is an extension of the two-fluid problem with surface tension at the SS, the lower fluid being of finite depth. Mandal (1981) discussed the case when the two fluids are of infinite extent both horizontally and vertically, the surface tension effect being taken into account.

2. Statement and formulation of the problem : We consider the irrotational motion of two non-viscous incompressible fluids under the action of gravity, considering the effect of surface tension at the surface separating the two fluids. The lower fluid is of infinite horizontal extent bounded by a horizontal bottom at some constant depth h, while the upper fluid is unbounded. The motion is time-harmonic with a small amplitude and angular frequency σ and due to an oscillatory singularity in one of the fluids. In each case the velocity potentials of the lower and upper fluids are simple harmonic with period. $2\pi/\sigma$. We use complex-valued potentials $\phi_1 e^{-i\sigma t}$, $\phi_2 e^{-i\sigma t}$ of which the actual velocity potentials are real parts. The velocity potentials in the two mediums are obtained when the singularities are present in the different mediums.

We take origin O in the mean SS and the axis Oy pointing vertically downwards into the lower fluid. Let ϕ_1 and ϕ_2 respectively denote the velocity potentials in the lower and upper fluid. We choose the y axis in such a way that it passes through the singularity, so that the point where the velocity potential has a singularity is $(0, \eta)$ or $(0, -\eta)$ according as the singularity is in the lower or upper fluid respectively (cf. Fig. 1).

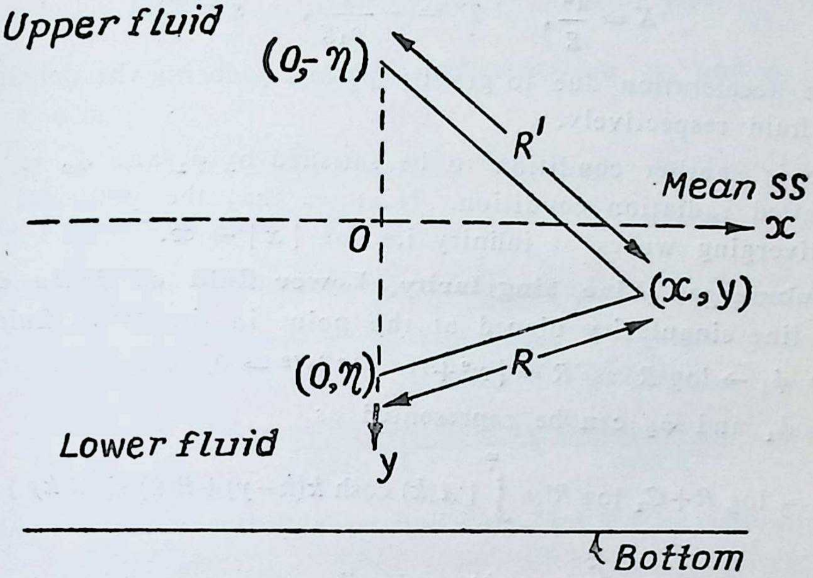


Fig. 1.

Then,

$$\nabla^2 \phi_1 = 0, \qquad 0 \leqslant y \leqslant h,$$

$$\nabla^2 \phi_2 = 0, \qquad y \leq 0$$

while $|x| < \infty$ except at the point of singularity which is taken as $(0, \eta)$ or $(0, -\eta)$.

Also,

$$\left.\begin{aligned} \partial\phi_1/\partial y &= 0 \qquad \text{on} \qquad y = h \\ \nabla\phi_2 &= 0 \qquad \text{as} \qquad y \to -\infty \end{aligned}\right\} \tag{2.1}$$

and at the SS, due to continuity of velocity

$$\left.\frac{\partial\phi_1}{\partial y}\right|_{y=+0} = \left.\frac{\partial\phi_2}{\partial y}\right|_{y=-0}. \tag{2.2}$$

Due to the presence of surface tension at the SS of the two fluids, the pressure on the two sides is discontinuous by the amount T/R where T is the surface tension between the two fluids and R is the radius of curvature of the *SS*. Neglecting the quantities of higher order of smallness the pressure condition at the *SS* gives

$$\left(K\phi_1+\frac{\partial\phi_1}{\partial y}\right)_{y=+0} - s\left(K\phi_2+\frac{\partial\phi_2}{\partial y}\right)_{y=-0} = T'\left(\frac{\partial^2\phi_1}{\partial y^2}\right)_{y=+0} \tag{2.3}$$

where

$$K=\frac{\sigma^2}{g}, \qquad T'=-\frac{T}{\rho_1 g}, \qquad s=\frac{\rho_2}{\rho_1},$$

g being the acceleration due to gravity ρ_1 and ρ_2 being the densities of the lower and upper fluid respectively.

There is another condition to be satisfied by ϕ_1 and ϕ_2 as $|x|\to\infty$ which is the so-called radiation condition. It states that the potential function should represent diverging waves at infinity i.e., as $|x|\to\infty$.

3. **Submerged Line Singularity, Lower fluid of finite depth :** We first consider a line singularity placed at the point in the lower fluid of depth h.

Then $\phi_1 \to \log R$ as $R=\{x^2+(y-\eta)^2\}^{1/2}\to 0$.

Now ϕ_1 and ϕ_2 can be represented as

$$\phi_1=\log R+C_1\log R'+\int_0^\infty\{A(k)\cosh k(h-y)+B(k)\sinh ky\}\cos kx\,dk \tag{3.1}$$

$$\phi_2=C_2\log R+\int_0^\infty C(k)\,e^{ky}\cos kx\,dk \tag{3.2}$$

where

$$R'=\{x^2+(y+\eta)^2\}^{1/2} \qquad \text{and} \qquad C_1,\ C_2,\ A(k),\ B(k)$$

and $C(k)$ are to be found from the conditions (2.1), (2.2) and (2.3) and also the condition that the integrals are to be convergent. The radiation condition will be dealt with in the sequel.

The following integral representations will be needed in our calculations

$$\frac{\partial}{\partial y}\log R=\int_0^\infty e^{-k(y-\eta)}\cos kx\,dk, \quad y>\eta,$$

$$=-\int_0^\infty e^{-k(\eta-y)}\cos kx\,dk, \quad y<\eta,$$

$$\frac{\partial}{\partial y}\log R'=\int_0^\infty e^{-k(y+\eta)}\cos kx\,dk, \quad y>-\eta$$

$$=-\int_0^\infty e^{k(y+\eta)}\cos kx\,dk, \quad y<-\eta$$

From the conditions (2.1), (2.2) and (2.3) we obtain

$$B = \frac{-e^{-kh}\,(e^{k\eta}+C_1\,e^{-k\eta})}{k}\,\text{sech}\;kh \tag{3.3}$$

$$\begin{aligned} k(A\sinh kh+C) &= (C_1-1+C_2)\,e^{-k\eta}+kB \\ &= F(k), \quad \text{say} \end{aligned} \tag{3.4}$$

$$1+C_1 = s\,C_2 \tag{3.5}$$

and

$$KA\cosh kh+\{\,(1-s)\,k-sK-T'k^3\}\;C = (1-s-T'k^2\;\; C_2\,e^{-k\eta} \tag{3.6}$$

Now for convergence of the integral representation ϕ_1 and ϕ_2 we must have $F(k)=0$ as $k \to 0$.

Hence $C_1 = 2s-1$ and $C_2 = 2$.

Solving (3.4) and (3.6) we obtain

$$\begin{aligned} A(k) = {} & \frac{2}{\Delta(k)}\Big[\,\delta(k)\cosh k(h-\eta)\,\Big]+\frac{2s}{k\cosh kh}\Big[\,e^{-k\eta}-\delta(k)\cosh k(h-\eta)\,\Big] \\ & +\frac{2k^2T'}{(1-s)\,\Delta(k)}\Big[\frac{se^{-k\eta}\sinh kh}{\cosh kh}+\frac{e^{-kh}}{\delta(k)\cosh kh} \\ & +\frac{s\delta(k)\sinh kh\cosh k(h-\eta)}{\mathrm{eosh}\;kh}-\frac{\cosh k(h-\eta)}{\cosh kh}-e^{-k\eta}\,\Big] \end{aligned} \tag{3.7}$$

$$\begin{aligned} C(k) = {} & -\frac{2\delta(k)}{\Delta(k)}\cosh k\,(h-\eta)\sinh kh+\frac{2}{k}\Big[\,e^{-k\eta}-\delta(k)\cosh k(h-\eta)\,\Big] \\ & -\frac{2k^2T'}{(1-s)\,\Delta(k)}\Big[\frac{se^{-k\eta}\sinh kh}{\cosh kh}+\frac{e^{-k\eta}}{\delta(k)\cosh kh} \\ & +\frac{s\delta(k)\sinh kh\cosh k(h-\eta)}{\cosh kh}-\frac{\cosh k\,(h-\eta)}{\cosh kh}-e^{-k\eta}\,\Big] \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} \Delta(k) &= \frac{1}{1-s}\Big[\,K\cosh kh-\{\,(1-s)\,k-sK-T'k^3\,\}\sinh kh\,\Big] \\ &= \frac{K}{1-s}\,(\cosh kh+s\sinh kh)-k\sinh kh+\frac{T'}{1-s}\,k^3\sinh kh \\ &= \frac{1}{2}\Big[\,k+K-\frac{T'k^3}{1-s}\,\Big]e^{-kh}-\frac{1}{2}\Big[\,k-\frac{1+s}{1-s}\,K-\frac{T'k^3}{1-s}\,\Big]e^{kh} \end{aligned} \tag{3.9}$$

$$\frac{1}{\delta(k)} = \cosh kh+s\sinh kh \tag{3.10}$$

$$F(k) = 2\left\{\frac{e^{-k\eta}}{\delta(k)}-\cosh k(h-\eta)\right\}\text{sech}\;kh. \tag{3.11}$$

Also,

$$\frac{K}{1-s}\cdot\frac{1}{k\Delta(k)} = \delta(k)\Big[\,\frac{\sinh kh}{\Delta(k)}+\frac{1}{k}-\frac{k^2T'}{(1-s)\,\Delta(k)}\;\sinh kh\,\Big] \tag{3.12}$$

Now $\Delta(k)$ has one simple pole at $k=k_0$, say, on the real axis of k. Below this pole we make an indentation of the contours of integration. Thus using the above results we get

$$\phi_1 = \log R+(2s-1)\log R' - \int_0^{\infty} \frac{2}{k}\Bigg[\frac{s\delta \cosh k\, h-\eta)\cosh k(h-y)}{\cosh kh} + \frac{e^{-kh}\sinh k\eta \sinh ky}{\cosh kh} - s\, e^{-k(h+\eta)}\Bigg]\cos kx\, dk$$

$$+2\mathop{\mathfrak{F}}_0^{\infty}\frac{1}{\Delta}\Bigg[\delta\cosh k(h-\eta)+\frac{T'k^2}{1-s}\Bigg\{\frac{s\, e^{-k\eta}\sinh kh}{\cosh kh}+\frac{e^{-k\eta}}{\delta\cosh kh}$$

$$+\frac{s\delta\sinh kh\cosh k(h-\eta)}{\cosh kh}-\frac{\cosh k(h-\eta)}{\cosh kh}-e^{-k\eta}\Bigg\}\Bigg]\cosh k(h-y)\cos kx\, dk \tag{3.13}$$

$$\phi_2 = 2\log R+\int_0^{\infty}\frac{2}{k}\Big\{e^{-k\eta}-\delta\cosh k(h-\eta)\Big\}e^{ky}\cos kx\, dk$$

$$-2\mathop{\mathfrak{F}}_0^{\infty}\frac{1}{\Delta}\Bigg[\delta\cosh k(h-\eta)+\frac{T'k^2}{1-s}\Bigg\{\frac{se^{-k\eta}\sinh kh}{\cosh kh}+\frac{e^{-kh}}{\delta\cosh kh}$$

$$+\frac{s\delta\sinh kh\cosh k(h-\eta)}{\cosh kh}-\frac{\cosh k(h-\eta)}{\cosh kh}-e^{-k\eta}\Bigg\}\Bigg]e^{ky}\sinh kh\cos kx\, dk. \tag{3.14}$$

Putting $T'=0$ we find that the results coincide with that of Gorgu and Kassem (1978).

Putting $s=0$ and $T'=-M$ (notation of Rhodes–Robinson (1970)) we get

$$\phi_1 = \log\frac{R}{R'}-2\mathop{\mathfrak{F}}_0^{\infty}\Bigg[\frac{(1+Mk^2)\cosh k(h-\eta)\cosh k(h-y)}{k(1+Mk^2)\sinh kh - K\cosh kh} + \frac{e^{-kh}\sinh k\eta\sinh ky}{k}\Bigg]\frac{\cos kx}{\cosh kh}\, dk.$$

which is the result derived by Rhodes-Robinson (1970).

Now to investigate the behavior of the integrals for large $|x|$, we put $2\cos x = e^{ik|x|}+e^{-ik|x|}$.

Then,

$$2\mathop{\mathfrak{F}}_0^{\infty}\frac{1}{\Delta}\Bigg[\delta\cosh k(h-\eta)+\frac{T'k^2}{1-s}\Bigg\{\frac{se^{-k\eta}\sinh kh}{\cosh kh}+\frac{e^{-k\eta}}{\delta\cosh kh}$$

$$-\frac{s\delta\sinh kh\cosh k(h-\eta)}{\cosh kh}-\frac{\cosh k(h-\eta)}{\text{eosh}\, kh}-e^{-k\eta}\Bigg\}\Bigg]\cosh k(h-y)\cos kx\, dk$$

$$=\mathop{\mathfrak{F}}_0^{8} I(k)\, e^{ik|x|}\, dk+\mathop{\mathfrak{F}}_0^{\infty} I(k)\, e^{-ik|x|}\, dk, \quad \text{(say)} \tag{3.15}$$

where

$$I(k)=\frac{\begin{array}{l}2\,[\,(1-s)\cosh kh\cosh k(h-\eta)+\\ T'k^2\{2s^2\,e^{-k\eta}\sinh^2 kh+se^{-k\eta}\sinh 2kh-\cosh kh\cosh k(h-\eta)\}]\cosh k(-y)\end{array}}{\cosh kh\,(\cosh kh+s\sinh kh)\,[K\cosh kh+\{sK-(1-s)\,k+T'k^3\}\sinh kh\,]}\,.$$

For the first integral of (3.15) we consider a contour in the quadrant bounded by the real axis of large length R_1 with an indentation below the pole $k=k_0$, a quarter of a circle with radius R_1 and the positive imaginary axis of length R_1.

The singularities of the integral are poles at k_0 and at $k_n=\alpha_n+i\beta_n$ where $\beta_n>0$. We then obtain

$$\fint_0^{R_1} I(k)\,e^{ik|x|}\,dk+\int_{C_{R_1}} I(k)\,e^{ik|x|}\,dk-i\int_0^{R_1} I(ik)\,e^{-k|x|}\,dk$$

$$=2\pi i\,[\text{ Residue of } I(k) \text{ at } k=k_0+\sum \text{Residue of } I(k) \text{ at } (\alpha_n+i\beta_n)\,] \qquad (3.16,$$

As the radius R_1 becomes large, the second integral in the left side becomes small. Now the residue at $(\alpha_n+i\beta_n)$ will give a term which has a factor $e^{i\alpha_n|x|-\beta_n|x|}$ which tends to zero as $|x|\to\infty$.

Hence making $R_1\to\infty$ we find that as $|x|\to\infty$

$$\fint_0^{\infty} I(k)\,e^{ik|x|}\,dk\to 2\pi i\text{ residue of } I(k)\text{ at } k=k_0 \qquad (3.17)$$

For the second integral of (3.15) we consider a contour in the fourth quadrant by the real axis of large length R_1 with an indentation below $k=k_0$, by a quarter of a circle with radius R_1 and the negative imaginary axis of length R_1. Since there is no singularity within the contour and the integrals along the negative imaginary axis and the quarter of the circle tend to zero as the radius tends to infinity and $|x|\to\infty$, we obtain that as

$$|x|\to\infty.\quad \fint_0^{\infty} I(k)\,e^{-ik|x|}\,dk\to 0.$$

Hence we find that as $|x|\to\infty$, ϕ_1 tends to

$$\frac{\begin{array}{l}2\pi i\,[\,(1-s)\cosh k_0(h-\eta)+T'k_0^2\,\{\,2s^2\,e^{-k_0\eta}\sinh^2 k_0h\\ \quad+se^{-k_0\eta}\sinh 2k_0h-\cosh k_0h\cosh k_0(h-\eta)\,\}\,]\cosh k_0(h-y)\,e^{ik_0|x|}\end{array}}{\begin{array}{r}\cosh k_0h\,(\cosh k_0h+s\sinh k_0h)\,[\,\{\,hK-(1-s)+3T'k_0^2\,\}\sinh k_0h\\ +\{\,sk-(1-s)\,k_0+T'k_0^3\,\}\,h\cosh k_0h\,]\end{array}} \qquad (3.18)$$

Putting $T' = 0$ and $M = (1+s)/(1-s)K$ according to the notation of Gorgu and Kassem (1978), (3.18) reduces to

$$\frac{2\pi i M\,[\,e^{-k_0(h-\eta)}+e^{k_0(h-\eta)}]}{(1+s)\,[\,(2hM-2k_0h-1)\,e^{k_0h}+e^{-k_0h}}\cdot\frac{\cosh k_0(h-y)}{k_0\sinh k_0h}\;e^{ik_0|x|}$$

which coincide with the result of Gorgu and Kassem (1978). Putting $s = 0$, $T' = -M$ (notation of Rhodes-Robinson (1970)) and $K = k_0(1+Mk_0^2)\tanh k_0h$ (3.18) reduces to

$$-\,4\pi i\,\frac{(1+Mk_0^2)\cosh k_0(h-n)\cosh k_0(h-y)\,e^{ik_0|x|}}{2k_0h(1+Mk_0^2)+(1+3Mk_0^2)\sinh 2k_0h}$$

which is the result derived by Rhodes-Robinson (1970).

Similarly, as $|x| \to \infty$, ϕ_2 tends to

$$\frac{\begin{array}{r}-2\pi i\,[\,(1-s)\cosh k_0h\cosh k_0\,(h-\eta)+T'k_0^2\{\,e^{-k_0\eta}\,(s\sinh^2 k_0h-\cosh^2 k_0h)\\ +\,e^{-k_0h}\,(\cosh k_0h+s\sinh k_0h)^2-\cosh k_0h\cosh k_0(h-\eta)\}]\\ \times e^{k_0y}\sinh k_0h\,e^{ik_0|x|}\end{array}}{\begin{array}{r}\cosh k_0h\,(\cosh k_0h+s\sinh k_0h)\,[\{hk-(1-s)+3T'k_0^2\}\sinh k_0h\\ +\,\{sk-(1-s)\,k_0+T'k_0^3\}\,h\cosh k_0h]\end{array}} \tag{3.19}$$

Putting $T'=0$ and $M=(1+s)/(1-s)K$, (3.19) reduces to

$$-\frac{2\pi i M\,[\,e^{-k_0(h-\eta)}+e^{k_0(h-\eta)}\,]}{(1+s)\,[2hM-2k_0h-1)\,e^{k_0h}+e^{-k_0h}}\cdot\frac{e^{k_0y}}{k_0}\cdot e^{ik_0|x|}$$

which is the same as obtained by Gorgu and Kassem (1978).

4. Wave source submerged in upper fluid : In this case there is a logarithmic type singularity at the point $(0, -\eta)$. Here, $\phi_2 \to \log R$ as $R \to 0$ where now

$$R = \{\,x^2+(y+\eta)^2\}^{1/2}.$$

Now again ϕ_1 and ϕ_2 can be represented as

$$\phi_1 = c_1\log R+\int_0^\infty [\,A(k)\cosh k(h-y)+B(k)\sinh ky\,]\cos kx\,dk \tag{4.1}$$

$$\phi_2 = \log R+c_2\log R'+\int_0^\infty C(k)\,e^{ky}\cos kx\,dk. \tag{4.2}$$

Proceeding as in the previous case we obtain from (2.1), (2.2) and (2.4)

$$B = -(C/k)\,e^{-k(h+\;\;)}\operatorname{sech} kh \tag{4.3}$$

$$k\,(A\sinh kh+C)=(c_1-1+c_2)\,e^{-k\eta}+kB$$

$$=F_1(k),\qquad \text{say} \tag{4.4}$$

$$c_1=s(1+c_2) \tag{4.5}$$

and

$$[\,K\cosh kh-k(1-s)\sinh kh+T'k^3\sinh kh\,]\,A-sKC$$
$$=-(1-s)\,(c_1\,e^{-k\eta}+kB)+T'k^2\,(c_1\,e^{-k\eta}+kB) \tag{4.6}$$

For convergence $F_1=0$ at $k=0$ so that

$$c_2=1,\qquad c_1=2s. \tag{4.7}$$

Solving (4.4) and (4.6) we obtain

$$A(k)=\frac{e^{-k\eta}}{k}\left[\operatorname{sech} kh-\delta\right]-\frac{2s\,e^{-k\eta}\,\delta\sinh kh}{\Delta}\left[1+\frac{T'k^2}{1-s}\right] \tag{4.8}$$

$$C(k)=2s\delta\,e^{-k\eta}\sinh kh\left(\frac{\sinh kh}{\Delta}+\frac{1}{k}\right)-\frac{2T'k^2\,s\delta\,e^{-k\eta}\sinh^2 kh}{(1-s)\,\Delta} \tag{4.9}$$

where $\delta(k)$, $\Delta(k)$ are given previously. Hence,

$$\phi_1=2s\log R+2s\int_0^\infty\frac{e^{-k\eta}}{k}\,\{\,e^{-ky}-\delta\cosh k(h-\eta)\,\}\cos kx\,dk$$
$$+\int_0^\infty\frac{2s\delta\,e^{-k\eta}}{\Delta}\left[-1+\frac{T'k^2}{1-s}\right]\sinh kh\cosh k(h-y)\cos kx\,dk \tag{4.10}$$

$$\phi_2=\log RR'+2s\int_0^\infty\frac{\delta}{R}\,\sinh kh\,e^{-k(\eta-y)}\cos kx\,dx$$
$$+\int_0^\infty\frac{2s\delta\sinh^2 kh\,e^{-k(\ -y)}}{\Delta}\left[1-\frac{T'k^2}{1-s}\right]\cos kx\,dk \tag{411}$$

Proceeding similarly as in the previous case we see that the above potentials have the following forms as $|x|\rightarrow\infty$.

$$\frac{2\pi is\,e^{-k_0\eta}\sinh k_0h\cosh k_0\,(h-y)\,[-(1-s)+k_0^2T'\,]\,e^{ik_0|x|}}{(\cosh k_0h+s\sinh k_0h)\,[\,\{\,hk-(1-s)+3T'k_0^2\,\}\sinh k_0h+\{sk-(1-s)\,k_0+T'k_0^3\,\}\,h\cosh k_0h\,]} \tag{4.12}$$

and

$$\frac{2\pi is\,\sinh^2 k_0h\,e^{-k_0(\eta-y)}\,[\,(1-s)-k_0^2\,T'\,]\,e^{ik_0|x|}}{(\cosh k_0h+s\sinh k_0h)\,[\,\{hk-(1-s)+3T'k_0^2\,\}\sinh k_0h+\{\,sk-(1-s)k_0+T'k_0^3\,\}\,h\cosh k_0h} \tag{4.13}$$

so that they indeed represent outgoing waves.

Putting $T' = 0$, $M = (1+s)/(1-s)K$ we obtain from (4.12) and (4.13)

$$-\frac{4\pi i\, sM}{1+s} \cdot \frac{e^{-k_0\eta}}{(2hM-2k_0h-1)\, e^{k_0h}+e^{-k_0h}} \cdot \frac{\cosh k_0\,(h-y)}{k_0}\; e^{ik_0\,|\,x\,|}$$

and

$$\frac{4\pi i\, sM}{1+s} \cdot \frac{e^{-k_0\eta}\sinh\, k_0h}{(2hM-2k_0h-1)\, e^{k_0h}+e^{-k_0h}} \quad \frac{e^{k_0y}}{k_0}\; e^{ik_0\,|\,x\,|}$$

which coincide with the results of Gorgu and Kassem (1978).

5. Submerged Point Singularities. Lower fluid of finite depth :

Here $\phi_1 \sim \dfrac{P_n\,(\cos\zeta)}{R^{n+1}}$ as $R=\{r^2+(y-\eta)^2\}^{1/2} \to 0,\ n=0,\,1,\,2,\ldots$

where r is the distance from the y-axis.

We assume

$$\phi_1=\frac{P_n\,(\cos\zeta)}{R^{n+1}}+\int_0^\infty \{A\,(k)\cosh\, k\,(h-y)+B\,(k)\sinh\, k\,y\}\, J_0\,(kr)\, dk \tag{5.1}$$

$$\phi_2=\int_0^\infty C(k)\, e^{ky}\, J_0\,(kr)\, dk \tag{5.2}$$

We use the following representations

$$\frac{P_n\,(\cos\zeta)}{R^{n+1}}=\frac{1}{n\,!}\int_0^\infty k^n\, e^{-k(y-\eta)}\; J_0(kr)\, dk, \quad y>\eta$$

$$=\frac{(-1)^n}{n\,!}\int_0^\infty k^n\, e^{k(y-\eta)}\, J_0\,(kr)\, dk, \quad y<\eta$$

From conditions (2.1), (2.2) and (2.4) we obtain

$$B=\frac{k^n}{n\,!}\; e^{-k(h-\eta)}\,\text{sech}\; kh \tag{5.3}$$

$$A\,\sinh\; kh+C=B+\frac{(-1)^n}{n\,!}\; k^n\, e^{-k\eta} \tag{5.4}$$

$$-KA\,\cosh\; kh+\{sK-(1-s)\,k+T'k^3\}\, C=\frac{(-1)^n}{n\,!}\; Kk^n\, e^{-k\eta} \tag{5.5}$$

Solving these equations and substituting we obtain

$$\phi_1=\frac{P_n(\cos\zeta)}{R^{n+1}}-\int_0^\infty \frac{k^n}{n\,!\,\Delta}\Big[(-1)^n e^{-k\eta}\,\Big\{k+K-\frac{T'k^3}{1-s}\Big\}\cosh\, k\,(h-y)+$$
$$+e^{-k(h-\eta)}\,\Big\{\Big(k-\frac{1+s}{1-s}K\Big)\cosh\, ky+\frac{K}{1-s}\; e^{-ky}-\frac{T'k^3}{1-s}\cosh\, ky\Big\}\Big]\, J_0\,(kr)\, dk \tag{5.6}$$

$$\phi_2 = \frac{K}{(1-s)\,n!}\int_0^\infty \frac{k^n}{\Delta}\,\{e^{-k(h-\eta)} + (-1)^n e^{k(h-\eta)}\}\,e^{ky} J_0(kr)\,dk -$$

$$-\frac{T'}{(1-s)\,n!}\int_0^\infty \frac{k^{n+3}}{\Delta}\,\{e^{-k(h-\eta)}\tanh kh + (-1)^n e^{-k\eta}\sinh kh\}\,e^{ky} J_0(kr)\,dk \tag{5.7}$$

Putting $T' = 0$ and $M = (1+s)/(1-s)\,K$ the above results coincide with those of Gorgu and Kassem (1978).

Again putting $s = 0$ and $T' = -M$ in (5.6) we obtain

$$\phi_1 = \frac{P_n(\cos\zeta)}{R^{n+1}} + \frac{1}{n!}\int_0 \frac{k^n}{k(1+Mk^2)\ \sinh kh - K\cosh kh}\ .$$

$$.\ [(-1)^n e^{-k\eta}\{k(1+Mk^2)+K\}\ \cosh k(h-y)$$

$$+ e^{-k(h-\eta)}\{k(1+Mk^2)\ \cosh\ ky - K\ \sinh\ ky\}]\,J_0(kr)\,dk$$

which is the result derived by Rhodes-Robinson (1970).

Now by putting $2J_0(kr) = H_0^{(1)}(kr) + H_0^{(2)}(kr)$ and rotating the contour in the integral involving $H_0^{(1)}(kr)$ in the first quadrant and in the integrals involving $H_0^{(2)}(k_r)$ in the fourth quadrant we can reduce the integrals in ϕ_1 and ϕ_2 into suitable forms from which it s seen that as $r \to \infty$, ϕ_1 and ϕ_2 respectively take the following forms

$$\frac{\begin{aligned}-\pi i(1-s)\,k_0^n\Big[(-1)^n e^{-k_0\eta}\Big\{k_0 + K - \frac{T'k_0^3}{1-s}\Big\}\cosh k_0(h-y) +\\ + e^{-k_0(h-\eta)}\Big\{k_0 - K.\,\frac{1+s}{1-s}\Big\}\ \cosh k_0 y +\\ + \frac{K}{1-s}\,e^{-k_0 y} - \frac{T'k_0^3}{1-s}\ \cosh k_0 y\Big\}\Big]\,H_0^{(1)}(k_0 r)\end{aligned}}{n!\,[\{hK-(1-s)+3T'k_0^2\}\ \sinh k_0 h + \{sK - (1-s)\,k_0 + T'k_0^3\}\,h\ \cosh k_0 h]}$$

and (5.8)

$$\frac{\begin{aligned}\pi i\,[Kk_0^n\{e^{-k_0(h-\eta)} + (-1)^n e^{k_0(h-\eta)}\}e^{k_0 y} - T'\ \{k_0{}^{n+3}(e^{-k_0(h-\eta)}\tanh k_0 h +\\ + (-1)^n e^{-k_0\eta}\sinh\ k_0 h)\ e^{k_0 y}\}]\ H_0^{(1)}(k_0 r)\end{aligned}}{n!\,\{hK-(1-s)+3T'k_0^2\}\ \sinh k_0 h + \{sK - k_0(1-s) + T'k_0^3\}\,h\cosh k_0 h]} \tag{5.9}$$

Putting $T' = -M$, $s = 0$, $K = k_0(1+Mk_0^2)\ \tanh k_0 h$, (5.8) reduces to

$$\frac{1}{n!}\cdot\frac{2\pi i\,k_0^{n+1}\,(1+Mk_0^2)[(-1)^n e^{k_0(h-\eta)} + e^{-k_0(h-\eta)}]}{2k_0 h\,(1+Mk_0^2) + (1+3Mk_0^2)\ \sinh 2k_0 h}\,.\ \cosh k_0(h-y)\,H_0^{(1)}(k_0 r)$$

which is derived by Rhodes-Robinson (1970).

Again putting $T'=0$ and $M=(1+s)/(1-s)\ K$ (5.8) and (5.9) take respectively the following forms

$$-\frac{2\pi i M k_0^n\ [e^{-k_0(k-\eta)}+(-1)^n\ e^{k_0 h(h-\eta)}]}{n!\,(1+s)\,[(2hM-2k_0h-1)\,e^{k_0h}+e^{-k_0h}]}\cdot\frac{\cosh k_0(h-y)}{k_0 \sinh k_0 h}\ H_0^{(1)}(k_0r)$$

and

$$\frac{2\pi i M k_0^n\ [e^{-k_0(h-\eta)}+(-1)^n e^{k_0(h-\eta)}]\ e^{k_0y}\ H_0^{(1)}(k_0r)}{n!\,(1+s)\,[(2hM-2k_0h-1)\ e^{k_0h}+e^{-k_0h}]}$$

which were derived by Gorgu and Kassem (1978).

6. Multipole submerged in upper fluid : Here

$$\phi_1 \sim \frac{P_n(\cos\zeta)}{R^{n+1}}$$

as $R=[r^2+(y+\eta)^2]^{1/2} \to 0$. Proceeding similarly we obtain in this case

$$\phi_1=\frac{2sK}{(1-s)\,n!}\int_0^\infty \frac{k^n}{\Delta}\ e^{-k\eta}\cosh\ k\,(h-y)\,J_0(kr)\,dk \tag{6.1}$$

$$\phi_2=\frac{P_n(\cos\zeta)}{R^{n+1}}+\frac{1}{n!}\int_0^\infty\left(1-\frac{2sK}{1-s}\cdot\frac{\sinh kh}{\Delta}\right)k^n\,e^{k(y-\eta)}\,J_0(kr)\,dk \tag{6.2}$$

Putting $T'=0$ and $M=(1+s)/(1-s)\ K$ we find that the results coincide with those of Gorgu and Kassem (1978).

Proceeding similarly as in the previous cases we find that as $r \to \infty$, ϕ_1 and ϕ_2 respectively take the following forms

$$\frac{2\pi i K s k_0^n\ e^{-k_0\eta}\cosh k_0(h-y\ H_0^{(1)}(k_0r)}{n!\,[\{hK-(1-s)+3T'k_0^2\}\sinh k_0h+\{sK-(1-s)\,k_0+T'k_0^3\}\ h\cosh k_0h]}$$

and

$$\frac{-2\pi i K k_0^n s\ \sinh\ k_0h\ e^{k_0(y-\eta)}\ H_0^{(1)}(k_0r)}{n!\,[\{hK-(1-s)+3T'k_0^2\}\ \sinh k_0h+\{sK-(1-s)\,k_0+T'k_0^3\}\ h\cosh k_0h]}$$

Putting $T'=0$ and $M=(1+s/1-s)\ K$ the above forms respectively reduce to the results of Gorgu and Kassem (1978).

7. Conclusion : Integral representation of the potential functions in a two-layered fluid medium with surface tension in *SS* are obtained in the case when the lower fluid is of finite depth. The different results reduce to the expressions given by Gorgu and Kassem (1978) when we put $T'=0$ and to those given by Rhodes-Robinson (1970) when we put $s=0$ i.e., a single fluid medium with surface tension on the free surface.

Acknowledgement : I thank Dr. B. N. Mandal, Department of Applied Mathematics, Calcutta University, for his help during this investigation.

References

Gorgu, M. A. and Kassem S. E. (1978) : "Basic singularities in the theory of internal waves," *Quart. Journ. Mech. & Appl. Math.* **31**, 31.

Mandal, B. N. (1981) : "Some basic singularities in the theory of internal waves in a two-fluid medium in the presence of surface tension", J. Tech. **26**, 11.

Rhodes-Robinson, P. F. (1970) : "Fundamental singularities in the theory of water waves with surface tension," *Bull. Australian Math. Soc.* **2**, 317.

Throne, R. C. (1953) : "Multipole Expansion in the theory of surface waves," *Proc. Cambridge Phil. Soc.* **49**, 707.

NARIKELDANGA HIGH SCHOOL,
50, KAVI SUKANTA SARANI,
CALCUTTA-700085

Bull. Cal. Math. Soc., **75**, 283-287 (1983)

REAL INVERSION FORMULA FOR THE DISTRIBUTIONAL GENERALIZED LAGUERRE TRANSFROM

T. N. VERMA

(*Received 20 October 1978 ; Revision Received 18 November 1980*)

Abstract : In the present paper we have obtained a real inversion formula for the generalized Laguerre transform defined below by (1). Inversion formulae given earlier by Debnath (1960) and McCully (1960) follow as its particular cases.

Introduction : The object of this paper is to obtain an inversion formula for the generalized Laguerre transform of $f(t)$, denoted by $f_n^{(\alpha,\beta)}(s)$ and defined by

$$f_n^{(\alpha,\beta)}(s)=\int_0^\infty e^{-st}\,(st)^\alpha\,L_n^{(\beta)}(st)\,f(t)\,dt \qquad \dots\ (1)$$

where the integral on the right of (1) exists in the sense of Lebesgue or in particular $f(t)$ is integrable (L) in $(0,\infty)$; $L_n^{(\beta)}(x)$ being the generalized Laguerre polynomials of degree n and order β (n non-negative integer ; $\beta>-1$). The parameters $s\ (>0)$, α and β are all assumed to be real.

In fact, the inversion formula obtained here for the above defined transform (1) is

$$f(t)=s\,(s)^{\beta-\alpha}\sum_{n=0}^{\infty}\frac{n\,!}{\Gamma(1+\beta+n)}\,L_n^{(\beta)}(st)\,f_n^{(\alpha,\beta)}(s) \qquad \dots\ (2)$$

which is quite general and as its special cases, one may derive from it the following inversion formulae

$$f(t)=\sum_{n=0}^{\infty}\frac{n\,!}{\Gamma(1+\alpha+n)}\,L_n^{(\alpha)}(t)\int_0^\infty e^{-t}\,t^\alpha\,L_n^{(\alpha)}(t)\,f(t)\,dt \qquad \dots\ (3)$$

and

$$f(t)=\sum_{n=0}^{\infty}L_n(t)\int_0^\infty e^{-t}\,L_n(t)\,dt \qquad \dots\ (4)$$

given by Debnath (1960) and McCully (1960) respectively.

We now prove the following theorem which gives the inversion formula desired.

Theorem : *Let $f(t)$ be a distribution with its support contained in the interval $0 < t < \infty$ and let the generalized Laguerre transform of $f(t)$ denoted by $f_n^{(\alpha,\beta)}(s)$, be given by the integral*

$$f_n^{(\alpha,\beta)}(s) = \int_0^\infty e^{-st}\,(st)^\alpha\, L_n^{(\beta)}(st)\, f(t)\, dt$$

where the parameters $s\ (>0)$, α and $\beta\ (>-1)$ are all assumed to be real and n is a non-negative integer.

Then

$$f(t) = s(st)^{\beta-\alpha} \sum_{n=0}^{\infty} \frac{n\,!}{\Gamma(1+\beta+n)}\, L_n^{(\beta)}(st)\, f_n^{(\alpha,\beta)}(s).$$

Proof : We have

$$f_n^{(\alpha,\beta)}(s) = \int_0^\infty e^{-st}\,(st)^\alpha\, L_n^{(\beta)}(st)\, f(t)\, dt$$

Then

$$\sum_{n=0}^{\infty} z^{-n} f_n^{(\alpha,\beta)}(s) = \sum_{n=0}^{\infty} z^{-n} \int_0^\infty e^{-st}\,(st)^\alpha\, L_n^{(\beta)}(st)\, f(t)\, dt$$

$$= \int_0^\infty e^{-st}\,(st)^\alpha f(t)\, dt \sum_{n=0}^{\infty} z^{-n}\, L_n^{(\beta)}(st) \qquad \ldots\ (5)$$

{ Genin *et al*. 1968 }

But,

$$\sum_{n=0}^{\infty} L_n^{(\beta)}(x)\, t^n = \frac{1}{(1-t)^{1+\beta}}\, e^{-xt/1-t} \qquad \ldots\ (6)$$

{ Rainville (1960) }

Combining (5) and (6), we have the relation

$$\sum_{n=0}^{\infty} z^{-n} f_n^{(\alpha,\beta)}(s) = \left(\frac{z}{z-1}\right)^{1+\beta} \int_0^\infty e^{-stz/(z-1)}\,(st)^\alpha f(t)\, dt \qquad \ldots\ (7)$$

Furthermore, let

$$F(p) = \int_0^\infty e^{-pt}\,(st)^\alpha f(t)\, dt \qquad \ldots\ (8)$$

Then, by writing $\frac{sz}{z-1}$ for p in (8), we get

$$F\left(\frac{sz}{z-1}\right) = \int_0^\infty e^{-szt/(z-1)} (st)^\alpha f(t)\, dt$$

$$=\left(\frac{z-1}{z}\right)^{1+\beta} \sum_{n=0}^{\infty} z^{-n} f_n^{(\alpha,\beta)}(s) \text{ [by (7)]} \qquad \dots \quad (9)$$

Let us now put

$$\frac{sz}{z-1} = x \text{ so that } z^{-1} = 1 - \frac{s}{x}$$

Then, from (9) we have

$$F(x) = (s/x)^{1+\beta} \sum_{n=0}^{\infty} \left(1 - \frac{s}{x}\right)^n f_n^{(\alpha,\beta)}(s)$$

$$= s^{1+\beta} \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^r \binom{n}{r} s^r x^{-r-\beta-1} f_n^{(\alpha,\beta)}(s) \qquad \dots \quad (10)$$

where $\binom{n}{r}$ is the binomial co-efficient $\frac{n!}{(n-r)!\, r!}$

Now, differentiating (10) k-times (k is any positive integer) with respect to x, we get

$$f^{(k)}(x) = s^{1+\beta} \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^{k+r} \binom{n}{r} s^r (r+\beta+1)_k\, x^{-r-\beta-1-k} f_n^{(\alpha,\beta)}(s)$$

where $(r+\beta+1)_k = (r+\beta+1)(r+\beta+2) \dots (r+\beta+k)$.

Therefore,

$$\frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)$$

$$= \frac{1}{k!} s^{1+\beta} \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^r \binom{n}{r} s^r (r+\beta+1)_k \left(\frac{t}{k}\right)^{r+\beta} f_n^{(\alpha,\beta)}(s)$$

$$= \frac{s}{k!} \sum_{n=0}^{\infty} \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!\, r!} \cdot \frac{(st)^{r+\beta}}{k^{r+\beta}} \cdot \frac{\Gamma(k+r+\beta+1)}{\Gamma(r+\beta+1)} \cdot f_n^{(\alpha,\beta)}(s)$$

where

$$(r+\beta+1)_k=\frac{\Gamma(k+r+\beta+1)}{\Gamma(r+\beta+1)}\ (\beta>-1)\ \{\text{Rainville [1960, } p.\ 23(3)]\ \}$$

That is,

$$\lim_{k\to\infty}\frac{(-1)^k}{k\,!}\left(\frac{k}{t}\right)^{k+1}F^{(k)}\,(k/t)$$

$$=\lim_{k\to\infty}\sum_{n=0}^{\infty}\sum_{r=0}^{n}(-1)^r\,\frac{n\,!}{(n-r)\,!\,r\,!}\cdot\frac{s(st)^{r+\beta}}{\Gamma(r+\beta+1)}\cdot\frac{1}{k^{r+\beta}}$$

$$\cdot\frac{\Gamma(k+r+\beta+1)}{\Gamma(k+1)}f_n^{(\alpha,\beta)}(s)\ [\text{ since }\Gamma(k+1)=k\,!\,]$$

$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\frac{(-1)^r\,n\,!}{(n-r)\,!\,r\,!}\cdot\frac{s(st)^{r+\beta}}{\Gamma(r+\beta+1)}\lim_{k\to\infty}\left\{\frac{1}{k^{r+\beta}}\,\frac{\Gamma(k+r+\beta+1)}{\Gamma(k+1)}\right\}\cdot f_n^{(\alpha,\beta)}(s)$$

But

$$\frac{\Gamma(z+a)}{\Gamma(z+b)}\sim z^{a-b}\ (|\,z\,|\to\infty)$$

where a and b are any arbitrary constants. { Erdélyi, (1953,) vol. 1, p. 47 [4] }

So by virtue of this relation, we have in our case

$$\lim_{k\to 0}\frac{1}{k^{r+\beta}}\,\frac{\Gamma(k+r+\beta+1)}{\Gamma(k+1)}=1$$

Therefore,

$$\lim_{k\to\infty}\frac{(-1)^k}{k\,!}\,(k/t)^{k+1}\,F^{(k)}\,(k/t)$$

$$=s(st)^{\beta}\sum_{n=0}^{\infty}\frac{n\,!}{\Gamma(1+\beta+n)}\left\{\sum_{r=0}^{n}\frac{(-1)^r}{(n-r)\,!\,r\,!}\,\frac{(1+\beta)_n}{(1+\beta)_r}\,(st)^r\right\}f_n^{(\alpha,\beta)}\,(s)$$

$$=s(st)^{\beta}\sum_{n=0}^{\infty}\frac{n\,!}{\Gamma(1+\beta+n)}\,L_n^{(\beta)}\,(st)\,f_n^{(\alpha,\beta)}\,(s)\qquad\ldots\quad(11)$$

[Rainville (1960), p. 201 (3)]

But Zemanian 1966, p. 163) has shown distributionally that

$$\lim_{k\to\infty}\frac{(-1)^k}{k\,!}\,(k/t)^{k+1}\,F^{(k)}\,(k/t)=f\,(t)$$

Hence, from (8) and (11) we have,

$$f(t)\,(st)^{\alpha}=s(st)^{\beta}\sum_{n=0}^{\infty}\frac{n\,!}{\Gamma(1+\beta+n)}\,L_n^{(\beta)}(st)\,f_n^{(\alpha,\beta)}(s)$$

that is,

$$f(t)=s(st)^{\beta-\alpha}\sum_{n=0}^{\infty}\frac{n\,!}{\Gamma(1+\beta+n)}\,L_n^{(\beta)}(st)\,f_n^{(\alpha,\beta)}(s)$$

or,

$$f(t)=s(st)^{\beta-\alpha}\sum_{n=0}^{\infty}\frac{n\,!}{\Gamma(1+\beta+n)}\,L_n^{(\beta)}(st)\int_0^{\infty}e^{-st}\,(st)^{\alpha}\,L_n^{(\beta)}(st)\,.\,f(t)\,dt\;(0<t<\infty)$$

This completes the proof of the theorem.

Some special cases of the theorem proved above are noteworthy.

If $\alpha=\beta$, we get

$$f(t)=s\sum_{n=0}^{\infty}\frac{n\,!}{\Gamma(1+\alpha+n)}\,L_n^{(\alpha)}(st)\,f_n^{(\alpha)}(s),$$

which is the inversion formula for the transform

$$f_n^{(\alpha)}(s)=\int_0^{\infty}e^{-st}\,(st)^{\alpha}\,L_n^{(\alpha)}(st)\,f(t)\,dt.$$

For $\alpha=\beta$, and $s=1$, we get as special case Debnath's result [1960] stated in (3). Similarly, for $\alpha=\beta=0$, $s=1$, our result gives rise to one by McCully [1960] stated in (4).

Acknowledgement : The author is highly grateful to Dr. A. B. B. Lall, Department of Mathematics, Ranchi University for his keen interest in the preparation of this paper.

References

Debnath, L. (1960) : 'On Laguerre Transform', *Bull. Calcutta Math. Soc.*, **52**, 69.

Erdélyi, A. (1953) 'Higher Transcendental Functions' (H. T. F.), **I**, McGraw-Hill, New York.

Genin, Robert and Calvez, Leon-Claude, (1968) ; Sur Quelques Propriétes de la Transformation de Laguerre, *C. R. Acad. Sc.* Paris, Sér A-B 267, 108

McCully, J. (1960) : The Laguerre Transform, *SIAM REVIEW*, **2**, 3, 185.

Rainville, E. D. (1960) : Special Functions, Macmillan, New York.

Zemanian, A. H., (1966) : The Distributional Laplace and Mellin Transformations, *SIAM*, *J. Appl. Math.*, **14**, 41.

DEPARTMENT OF MATHEMATICS.
RANCHI AGRICULTURAL COLLEGE,
RANCHI, BIHAR (INDIA).

Bull Cal. Math. Soc., **75**, 289-294 (1983)

PROJECTIVE MOTION IN RECURRENT FINSLER SPACES

R. S. SINHA and V. S. P. CHOWDHURY

(*Received 10 June 1980*)

Introduction : Let F_n be an n-dimensional Finsler space with symmetric connection G^i_{jk}. Then an infinitesimal point transformation $\bar{x}^i = x^i + v^i(x) d\tau$ in F_n is called a projective motion if it transforms the system of geodesics or paths into the same system. The necessary and sufficient condition for this motion is (Sinha 1971)

$$\pounds\, G^i_{jk} = -\dot{\partial}_j P \delta^i_k - \dot{\partial}_k P \delta^i_j - \dot{x}^i \dot{\partial}_j \dot{\partial}_k P \qquad \ldots \quad (1.1)$$

where £ denotes the Lie derivative w.r.t. v^i, and $P(x, \dot{x})$ is a scalar funetion homogeneous of degree one in $\dot{x}^i$ and hence satisfies

$$\text{(a)} \quad \dot{x}^r \dot{\partial}_r P = P \quad \text{(b)} \quad \dot{x}^r \dot{\partial}_r \dot{\partial}_j P = 0. \qquad \ldots \quad (1.2)$$

Also F_n is called a recurrent Finsler space (Moor 1963) if its Berwald's curvature tensor H^i_{jkl} satisfies

$$H^i_{jkl(m)} = K_m(x, \dot{x}) H^i_{jkl}, \qquad \ldots \quad (1.3)$$

where (m) denotes covariant derivative w.r.t. x^m in Berwald's sense and $K_m(x, \dot{x})$ is a non-zero vector homogeneous of degree zero in $\dot{x}^i$. We shall denote such a space F^*_n in this paper and study the existence of projective motion in F^*_n.

2. Preliminaries :

We shall require the following identities (Rund 1959) in the course of discussion.

$$G^i_{rjk}\, \dot{x}^r = 0 \qquad \ldots \quad (2.1)$$

$$\text{(a)} \quad H^i_{jkh}\, \dot{x}j = H^i_{kh} \quad \text{(b)} \quad H^r_{jhr} = \dot{\partial}_j H_h = H_{jh} \quad \text{(c)} \quad H_{jk} \dot{x}^j = H_k$$

$$\text{(d)} \quad H_r\, \dot{x}^r = (n-1)H \quad \text{(e)} \quad H^i_{jkh} + H^i_{jhk} = 0 \quad \text{(f)} \quad H_{jk} \dot{x}^k = (n-1)\dot{\partial}_j H - H_j \qquad \ldots \quad (2.2)$$

In view of (1.3) the tensors appearing in (2.2) are recurrent.

Identities satisfied by projective curvature tensors :

$$\text{(a)} \quad W^i_{jkl} = \dot{\partial}_j W^i_{kl} \quad \text{(Generalised Weyl tensor)}$$

$$\text{(b)} \quad W^i_{kl} = W^i_{jkl}\,\dot{x}^j = H^i_{kl} + \frac{\dot{x}^i}{n+1}(H_{kl} - H_{lk}) + \frac{\delta^i_k}{n^2-1}(nH_l + \dot{x}^r H_{lr}) - \frac{\delta^i_l}{n^2-1}(nH_k + \dot{x}^r H_{kr})$$

$$\text{(c)} \quad W^i_{rk}\,\dot{x}^r = W^i_k \qquad \text{(d)} \quad W^i_{kl} + W^i_{lk} = 0 \qquad \text{(e)} \quad W^i_k\,\dot{x}^k = 0$$

$$\text{(f)} \quad W^r_{rkl} = 0 \qquad \text{(g)} \quad W^r_{kr} = 0 \qquad \ldots \quad (2\,3)$$

Commutation formulae :

$$\text{(a)} \quad T_{(j)(k)} - T_{(k)(j)} = -\dot{\partial}_r T H^r_{jk}$$

$$\text{(b)} \quad (\dot{\partial}_j T^i_k)_{(m)} - \dot{\partial}_j T^i_{k(m)} = -T^r_k G^i_{rjm} + T^i_r G^r_{kjm} \qquad \ldots \quad (2.4)$$

Differentiating (2.2) (b) covariantly *w.r.t.*x^m and using (1.3), (2.4) (b) and (2.2) (b), we have $\dot{\partial}_j K_m H_h + H_r G^r_{hjm} = 0$ which on transvecting by $\dot{x}^h$ and using (2.1) and (2.2) (d), gives $(n-1)\,\dot{\partial}_j K_m H = 0$. This shows that for F^*_n either

$$H = 0 \text{ if } \dot{\partial}_j K_m \neq 0 \qquad \ldots \quad (2.5)$$

or

$$\text{(a)} \quad \dot{\partial}_j K_m = 0 \text{ and (b)} \quad H_r G^r_{hjm} = 0. \qquad \ldots \quad (2.6)$$

Sinha (1971) has shown that for projective motion in F_n we have

$$\pounds\, H^i_{jkl} = \delta^i_j\,[(\dot{\partial}_l P)_{(k)} - (\dot{\partial}_k P)_{(l)}] + (\dot{\partial}_j P)_{(k)}\delta^i_l - (\dot{\partial}_j P)_{(l)}\delta^i_k + \dot{x}^i[(\dot{\partial}_j \dot{\partial}_l P)_{(k)} - (\dot{\partial}_j \dot{\partial}_k P)_{(l)}]$$

which, on contracting *i* and 1 and using (2.2) (b) and (1.2) (b) gives

$$\pounds\, H_{jk} = n(\dot{\partial}_j P)_{(k)} - (\dot{\partial}_k P)_{(j)} - \dot{x}^r(\dot{\partial}_j \dot{\partial}_k P)_{(r)}$$

on taking the Lie derivative of (2.3) (a), using the commutativity of the operations £ and $\dot{\partial}_j$ and introducing the preceding two equations, we have

$$\pounds\, W^i_{jkl} = 0. \qquad \ldots \quad (2.7)$$

3. **Motion in F^*_n for a Field v^i :** We now transvect (2.7) by $\dot{x}^j$ and differentiate (2.3) (b) covariantly *w.r.t.*x^m and obtain $\pounds\, W^i_{kl} = 0$ and $W^i_{kl(m)} = K_m W^i_{kl}$. On employing a formula (Yano 1955) of the type

$$\pounds\, T^i_{k(m)} - (\pounds T^i_k)_{(m)} = T^r_k(\pounds\, G^i_{rm}) - T^i_r(\pounds\, G^r_{km}) - \dot{\partial}_r T^i_k(\pounds\, G^r_{mh})\dot{x}^h$$

to W^i_{kl} and using the preceding two equations, (1.1), (2.3) (a) (b), (c), (d) and (1.2) (a), (b), we have

$$(£\, K_m - 2\dot{\partial}_m P)W^{i}_{kl} = -\delta^{i}_{m}\dot{\partial}_r PW^{r}_{kl} + \dot{\partial}_k PW^{i}_{ml} + \dot{\partial}_l PW^{i}_{km}$$
$$+PW^{i}_{mkl} - \dot{\partial}_r\dot{\partial}_m PW^{r}_{kl}\dot{x}^{i} + \dot{\partial}_k\dot{\partial}_m PW^{i}_{l} - \dot{\partial}_l\dot{\partial}_m PW^{i}_{k} \qquad (3.1)$$

which, on transvecting by $\dot{x}^m$ and using (1.2) (a), (b) and (2.3) (a), (b), (c), (d), gives

$$(£\, K_m\dot{x}^m - 3P)W^{i}_{kl} = -\dot{x}^{i}\dot{\partial}_r PW^{r}_{kl} + \dot{\partial}_k PW^{i}_{l} - \dot{\partial}_l PW^{i}_{k} \qquad \dots \quad (3.2)$$

Transvecting (3.2) by $\dot{x}^k$ and $\dot{\partial}_i P$ respectively and using (2.3) (c), (e) and (1.2) (a), we have

$$(£\, K_m\dot{x}^m - 4P)W^{i}_{l} = -\dot{x}^{i}\dot{\partial}_r PW^{r}_{l} \qquad \dots \quad (3.3)$$

and

$$(£\, K_m\dot{x}^m - 2P)\dot{\partial}_r PW^{r}_{kl} = \dot{\partial}_k P\dot{\partial}_r PW^{r}_{l} - \dot{\partial}_l P\dot{\partial}_r PW^{r}_{k} \qquad \dots \quad (3.4)$$

respectively. We may transvect (3.4) by $\dot{x}^k$ and use (2.3) (c), (e) and (1.2) (a) to obtain $(£\, K_m\dot{x}^m - 3P)\dot{\partial}_r PW^{r}_{l} = 0$. Suppose that $\dot{\partial}_r PW^{r}_{l} \neq 0$. Then the preceding equation gives

$$£\, K_m\dot{x}^m = 3P \qquad \dots \quad (3.5)$$

Contracting i and m in (3.1) and using (1.2) (b) and (2.3) (d), (e), (f), we have

$$[\,£\, K_m + (n-2)\dot{\partial}_m P]W^{m}_{kl} = \dot{\partial}_k\dot{\partial}_r PW^{r}_{l} - \dot{\partial}_l\dot{\partial}_r PW^{r}_{k}$$

which, on transvecting by $\dot{x}^k$ and using (2.3) (c), (e) and (1.2) (b), gives $[£\, K_m + (n-2)\dot{\partial}_m P]\, W^{m}_{l} = 0$. On multiplying the preceding equation by $(£\, K_r\dot{x}^r - 4P)$ and using (3.3) and (1.2) (a), we have $£\, K_m\dot{x}^m + (n-2)P = 0$ which, due to (3.5), gives $P = 0$. Thus we have the

THEOREM 3.1 : *The space F^*_n does not admit a projective motion satisfying the condition* $\dot{\partial}_r PW^{r}_{l} \neq 0$.

Theorem 3.1 asserts that for projective motion $\dot{\partial}_r PW^{r}_{l} = 0$ must hold, due to which (3.3) and (3.4) give $(£\, K_m\dot{x}^m - 4P)W^{i}_{l} = 0$ and $(£\, K_m\dot{x}^m - 2P)\dot{\partial}_r PW^{r}_{kl} = 0$. Since $W^{i}_{l} = 0$ implies $W^{i}_{jkl} = 0$, the preceding three equations give the

THEOREM 3.2 : *If the space F^*_n admits a projective motion for a field v^i then $\dot{\partial}_r PW^{r}_{l} = 0$ and one of the following two sets of conditions holds good :*

1. The generalised Weyl tensor vanishes.
2. $£\, K_m\dot{x}^m = 4P$ *and* $\dot{\partial}_r PW^{r}_{jk} = 0$.

4. The Recurrent Field : The recurrent field is characterised by the relation

$$v^{i}_{(j)} = \phi_j(x, \dot{x})v^{i} \qquad \dots \quad (4.1)$$

where $\phi^j(x, \dot{x}) \neq 0$ is a covariant vector homogeneous of degree zero in $\dot{x}^i$.

On employing (2.4) (b) to v^i and using (4.1) we have

$$G^i_{r\,j\,k} v^r = \dot{\partial}_j \phi_k v^i \qquad \ldots \quad (4.2)$$

On introducing the relation (Rund 1959)

$$\pounds\, G^i_{jk} = v^{\;i}_{(j)(k)} + H^{\;i}_{j\,k\,h} v^h + G^i_{j\,k\,h} v^{\;h}_{(r)} \dot{x}^r$$

in (1.1) and using (4.1) and (4.2) and (2.2) (e), we have

$$H^{\;i}_{j\,h\,k} v^h = (\phi_{j\,(k)} + \phi_j \phi_k + \phi_r \dot{x}^r \dot{\partial}_j \phi_k) v^i + \dot{\partial}_j P \delta^i_k + \dot{\partial}_k P \delta^i_j + \dot{x}^i \dot{\partial}_j \dot{\partial}_k P \qquad \ldots \quad (4.3)$$

Transvecting (4.3) by v^k and using (2.2) (e) we have

$$(\phi_{j\,(k)} + \phi_j \phi_k + \phi_r \dot{x}^r \dot{\partial}_j \phi_k) v^i v^k + v^i \dot{\partial}_j P + v^k \dot{\partial}_k P \delta^i_j + \dot{x}^i v^k \dot{\partial}_j \dot{\partial}_k P = 0. \qquad \ldots \quad (4.4)$$

Contracting i and j in (4.4) and using (1.2) (b) we have

$$(\phi_{j\,(k)} + \phi_j \phi_k + \phi_r \dot{x}^r \dot{\partial}_j \phi_k) v^j v^k + (n+1) v^j \dot{\partial}_j P = 0$$

Transvecting (4.4) by v^j and using the preceding equation we may obtain $(n-1) v^i v^j \dot{\partial}_j P = \dot{x}^i v^j v^k \dot{\partial}_j \dot{\partial}_k P$ which, on transvecting by $\dot{\partial}_i \dot{\partial}_l P$ and using (1.2) (b), gives

$$v^i \dot{\partial}_i \dot{\partial}_l P v^j \dot{\partial}_j P = 0.$$

The preceding two equations give

$$v^j \dot{\partial}_j P = 0, \qquad \ldots \quad (4.5)$$

due to which, (4.4) gives

$$(\phi_{j\,(k)} + \phi_j \phi_k + \phi_r \dot{x}^r \dot{\partial}_j \phi_k) v^k + \dot{\partial}_j P = 0. \qquad \ldots \quad (4.6)$$

Contracting i and k in (4.3) and using (2.2) (b), (4.6) and (1.2) (b) we have

$$H_{j\,h} v^h = n \dot{\partial}_j P. \qquad \ldots \quad (4.7)$$

which, on transvecting by $\dot{x}^j$ and using (2.2) (c) and (1.1) (a), gives

$$H_h v^h = nP. \qquad \ldots \quad (4.8)$$

On introducing (4.1) in the Ricci identity

$$v^{\;i}_{(j)(k)} - v^{\;i}_{(k)(j)} = H^{\;i}_{h\,j\,k} v^h + \dot{\partial}_r v^i H^{\;r}_{j\,k}$$

we have $A_{jk} v^i = H^{\;i}_{h\,j\,k} v^h$ where $A_{jk} = \phi_{j\,(k)} - \phi_{k\,(j)}$, whence on contracting i and k and using (2.2) (b) we have

$$H_{hj}v^h=A_{jh}v^h \qquad \ldots \quad (4.9)$$

Transvecting (4.3) by $\dot{\partial}_i P\dot{x}^j$ and using (2.2) (a), (4.5) and (1.2) (a) we have

$$\dot{\partial}_r PH^r_{hk}v^h=2P\dot{\partial}_k P. \qquad \ldots \quad (4.10)$$

Transvecting (2.3) (b) by $\dot{\partial}_i P\dot{x}kv^l$ and using (2.3) (c) (2.2) (a), (c), (e), (f), (4.10), (1.1) (a), (4.8) and (4.5), we have

$$\dot{\partial}_r PW^r_l v^l = \frac{n-2}{n-1}(P^2-P\dot{\partial}_l Hv^l)$$

which, on using theorem 3.2 and (2.5), gives $P=0$ if $n>2$ and $\dot{\partial}_j K_m \neq 0$. Thus we have the

THEOREM 4.1 : *The space* F^*_n $(n>2)$ *satisfying* $\dot{\partial}_j K_m \neq 0$ *does not admit projective motion for the recurrent field.*

Theorem 4.1 shows that for projective motion in F^*_n (2.6) (a), (b) must hold. Then transvecting (4.2) by H_i and using (2.6) (b), (4.8) and (4.2) we have

$$\text{(a)} \quad G^i_{rjk}v^r=0 \qquad \text{(b)} \quad \dot{\partial}_j\phi_k=0. \qquad \ldots \quad (4.11)$$

Let us assume that the conditions 2 of theorem 3.2 hold, which, on using the requisite expressions, (4.1), (2.6) (a) and (1.2) (a) give $\dot{x}^m(K_{m(l)}+K_l\phi_m)v^l=4P$ and

$$\dot{\partial}_r PH^r_{jk}+\frac{P}{n+1}(H_{jk}-H_{kj})+\frac{\dot{\partial}_j P}{n^2-1}(nH_k+\dot{x}^r H_{kr})-\frac{\dot{\partial}_k P}{n^2-1}(nH_j+\dot{x}^r H_{jr})=0$$

respectively. Differentiating the first of the preceding two equations partially $w.r.t.\dot{x}^k$ and using (2.4) (b), (2.1), (2.6) (a) and (4.11) (b) and transvecting the second by v^j and using (4.10), (4.7), (4.5) and (4.8) we have respectively

$$(K_{k(l)}+K_l\phi_k)v^l=4\dot{\partial}_k P \qquad \ldots \quad (4.12)$$

and

$$[(n-2)P-\dot{x}^r H_{jr}v^j]\dot{\partial}_k P+(n-1)PH_{jk}v^j=0$$

Transvecting the preceding equation by $\dot{x}^k$ and using (1.2) (a) we obtain $\dot{x}^r H_{jr}v^j=-P$ due to which, the same equation gives

$$H_{jk}v^j=-\dot{\partial}_k P \qquad \ldots \quad (4.13)$$

Differentiating (4.7) covariantly $w.r.t.x^k$ and using (4.1) and the same equation, we have

$$(K_k+\phi_k)\dot{\partial}_j P=(\dot{\partial}_j P)_{(k)} \qquad \ldots \quad (4.14)$$

Differentiating (4.14) covariantly $w.r.t.x^1$ and using itself we have an equation which gives

$$(K_{k(l)}-K_{l(k)}+A_{kl})\dot{\partial}_j P=(\dot{\partial}_j P)_{(k)(l)}-(\dot{\partial}_j P)_{(l)(k)}.$$

Transvecting the preceding equation by $\dot{x}^j v^l$ and using (1.2) (a), (4.9), (4.13), (2.4) (a), (2.2) (a), (e), and (4.10) we have $(K_{k(l)}-K_{l(k)})v^l=3\dot{\partial}_k P$ which, on subtracting from (4.12) and using (4.1), gives

$$(K_l v^l)_{(k)}=\dot{\partial}_k P \qquad \dots \quad (4.15)$$

From (4.15) we have $(\dot{\partial}_k P)_{(j)}-(\dot{\partial}_j P)_{(k)}=(K_l v^l)_{(k)(j)}-(K_l v^l)_{(j)(k)}$ Transvecting the preceding equation by v^k and using (2.4) (a), (2.6) (a), (4.5), (4.1) and (4.14) we have $(K_l+\phi_l)v^l=0$.

Introducing the preceding equation, (4.15) and (4.11) (b) in (4.6) we have $A_{jh}v^h=0$, due to which, (4.9) and (4.13) give a contradiction $\dot{\partial}_j P=0$. Thus the conditions 2 of theorem 3.2 can not hold. Then from the theorems 3.2, 4.1 and equations (2.6) (a), (b), (4.11) (a), we have the

THEOREM 4.2 : *If the space $F_n^*(n > 2)$ admits a projective motion for a recurrent field v^i, then the following conditions must hold good :*

1. *The recurrence vector of the space is a point function.*
2. *The generalised Weyl tensor of the space vanishes.*
3. $H_r G^r_{jkl}=0$.
4. $G^i_{rjk}v^r=0$.

References

Moor, A. (1963) : Unter-suchungen uber Finsler raume Von rekurrenter Krummung, *Tensor* (*N. S*), 13, 1.

Rund, H. (1959) : The differential geometry of Finsler spaces, *Springer-verlag*, Berlin.

Sinha, R. S. (1971) : Infinitesimal projective transformation in *Finsler spaces*, *Prog. Math* 5, 30.

Takano, K. (1962) : On projective motion in a space with recurrent curvature, *Tensor* (*N. S*), 12, 28.

Yamaguchi, S. (1967) : On infinitesimal projective transformation in non-Riemannian recurrent spaces, *Tensor* (*N. S.*), 18, 271.

Yano, K. (1955) : The theory of Lie derivatives and its applications, North Holland.

DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF GORAKHPUR,
GORAKHPUR (INDIA)
and
DEPARTMENT OF MATHEMATICS,
B. R. D. POST-GRADUATE COLLEGE,
DEORIA (INDIA).

Bull. Cal. Math. Soc., **75**, 295-302 (1983)

COMMUTATION FORMULAE FOR NON-LINEAR CONNECTIONS IN A FINSLER SPACE

R. S. SINHA and S. A. FARUQI

(*Received 3 October 1979*)

Abstract : This paper is devoted to the study of commutation formulae resulting from covariant differentiation in a Finsler space from the standpoint of non-linear connections.

1. Introduction : Non-linear connections have been studied by Kawaguchi (1952, 1956), Rund (1959), Kikuchi (1962) and many others. In the present paper some covariant derivatives have been defined with the help of non-linear connections and their commutation formulae have been established. The commutation formulae help in defining curvature tensors depending on non-linear connections. The notations of Rund (1959) have been used freely.

Consider an n-dimensional Finsler space F_n. Let X^i and Y_i be given contravarient and covariant differentiable vector fields. Let $\overset{1}{\Gamma}{}^i_j(x, X)$ and $\overset{2}{\Gamma}_{jk}(x, Y)$ be the two types of non-linear connection coefficients (1959). Let us put

$$\overset{1}{\Gamma}{}^i_{jk}(x, X)=\dot{\partial}_j \overset{1}{\Gamma}{}^i_k(x, X), \overset{2}{\Gamma}{}^i_{jk}(x, Y)=\dot{\partial}^i \overset{2}{\Gamma}_{jk}(x, Y), \qquad \dots \quad (1.1)$$

where $\dot{\partial}_j \equiv \frac{\partial}{\partial X^j}$, $\dot{\partial}^i \equiv \frac{\partial}{\partial Y_i}$

It is easy to verify that the transformation properties of $\overset{1}{\Gamma}{}^i_{jk}(x, X)$ and $\overset{2}{\Gamma}{}^i_{jk}(x, Y)$ are the same as those of Cartan's connection $\Gamma^{*1}_{jk}.(x, X)$.

For the sake of brevity, we shall now onwards denote $\overset{1}{\Gamma}{}^i_j(x, X)$, $\overset{2}{\Gamma}_{ij}(x, Y)$, $\overset{1}{\Gamma}{}^i_{jk}(x,X)$, $\overset{2}{\Gamma}{}^i_{jk}(x, Y)$, ... by $\overset{1}{\Gamma}{}^i_j$, $\overset{2}{\Gamma}_{ij}$, $\overset{1}{\Gamma}{}^i_{jk}$, $\overset{2}{\Gamma}{}^i_{jk}$,... respectively. We will also use the notation $\mathbf{G}_{jk}\{...\}$ which denotes, for instance, $\mathbf{G}_{jk}\{\wedge_{jk}\}=\wedge_{jk}-\wedge_{kj}$

Let $T^i_j(x, X)$, $\bar{T}^i_j(x, Y)$, $T^{i_1 \dots i_r}_{j_1 \dots j_s}(x, X)$ and $\bar{T}^{i_1 \dots i_r}_{j_1 \dots j_s}(x, Y)$ be arbitrary tensor fields. With the help of the two types of non-linear connections given by (1.1), we can define two types of covariant differentials as follows :

$$\overset{1}{\delta}\, T^i_j = d\, T^i_j + T^r_j\, \overset{1}{\Gamma}{}^i_{rl}\, dx^l - T^i_r\, \overset{1}{\Gamma}{}^r_{jl}\, dx^l, \qquad \cdots \quad (1.2)$$

$$\overset{2}{\delta}\, \bar{T}^i_j = d\, \bar{T}^i_j + \bar{T}^r_j\, \overset{2}{\Gamma}{}^i_{rl}\, dx^l - \bar{T}^i_r\, \overset{2}{\Gamma}{}^r_{jl}\, dx^l. \qquad \cdots \quad (1.3)$$

Consequently, we can define two types of covariant derivatives, which are as follows :

$$T^i_{j,k} = \partial_k\, T^i_j + \dot{\partial}_r\, T^i_j\, \partial_k\, X^r + T^r_j\, \overset{1}{\Gamma}{}^i_{rk} - T^i_r\, \overset{1}{\Gamma}{}^r_{jk}, \qquad \cdots \quad (1.4)$$

$$\bar{T}^i_{j,k} = \partial_k\, \bar{T}^i_j + \dot{\partial}^r\, \bar{T}^i_j\, \partial_k\, Y_r + \bar{T}^r_j\, \overset{2}{\Gamma}{}^i_{rk} - \bar{T}^i_r\, \overset{2}{\Gamma}{}^r_{jk}, \qquad \cdots \quad (1.5)$$

where $\partial_k \equiv \dfrac{\partial}{\partial x^k}$.

If we substitute $\partial_k X^r = -\overset{1}{\Gamma}{}^r_k$ and $\partial_k\, Y_r = \overset{2}{\Gamma}_{rk}$ (conditions for X^i and Y_i to be "stationary") in (1.4) and (1.5) respectively, the covariant derivatives (1.4) and (1.5) take the following forms :

$$T^i_{j|k} = \partial_k\, T^i_j - \dot{\partial}_r\, T^i_j\, \overset{1}{\Gamma}{}^r_k + T^r_j\, \overset{1}{\Gamma}{}^i_{rk} - T^i_r\, \overset{1}{\Gamma}{}^r_{jk} \qquad \cdots \quad (1.6)$$

$$\bar{T}^i_{j|k} = \partial_k\, \bar{T}^i_j + \dot{\partial}^r \bar{T}^i_j\, \overset{2}{\Gamma}_{rk} + \bar{T}^r_j\, \overset{2}{\Gamma}{}^i_{rk} - \bar{T}^i_r\, \overset{2}{\Gamma}{}^r_{jk} \qquad \cdots \quad (1.7)$$

By virtue of (1.6), (1.7), (1.4) and (1.5), we have

$$X^i_{|k} = 0, \;\; Y_{j|k} = 0,$$

$$T^i_{j,k} - T^i_{j|k} = \dot{\partial}_r\, T^i_j\, X^r_{,k},\; \bar{T}^i_{j,k} - \bar{T}^i_{j|k} = \dot{\partial}^r\, \bar{T}^i_j\, Y_{r;k}$$

Hence, from (1.2) and (1.3), we get

$$\overset{1}{\delta}\, T^i_j = T^i_{j,k}\, dx^k = \dot{\partial}_k\, T^i_j\, \overset{1}{\delta}\, X^k + T^i_{j|k}\, dx^k,$$

$$\overset{2}{\delta}\, \bar{T}^i_j = \bar{T}^i_{j;k}\, dx^k = \dot{\partial}^k\, \bar{T}^i_j\, \overset{2}{\delta}\, Y_k + \bar{T}^i_{j|k}\, dx^k.$$

With the help of (1.6), we can obtain the commutation, formula involving the operator $\dot{\partial}_h$ and the covariant differentiation (1.6), which is

$$(\dot{\partial}_h\, T^i_j)_{|k} - \dot{\partial}_h\, (T^i_{j|k}) = -T^r_j\, \dot{\partial}_h\, \overset{1}{\Gamma}{}^i_{rk} + T^i_r\, \dot{\partial}_h\, \overset{1}{\Gamma}{}^r_{jk},$$

and for the covariant differentiation (1.7), the commutation formula is

$$(\dot{\partial}^h\, \bar{T}^i_j)_{|k} - \dot{\partial}^h\, (\bar{T}^i_{j|k}) = -\bar{T}^r_j\, \dot{\partial}^h\, \overset{2}{\Gamma}{}^i_{rk} + \bar{T}^i_r\, \dot{\partial}^h\, \overset{2}{\Gamma}{}^r_{jk},$$

2. Commutation formulae for the vector fields X^i and Y_i : In this article, we will obtain the commutation formulae for the two types of covariant differentiation (1.4) and

(1.5) and will define the curvature tensors. According to definition (1.4), the covariant derivative of

$$X^{i}_{,j}=\partial_j X^{i}+X^{r}\overset{1}{\Gamma}{}^{i}_{rj} \text{ w. r. t. } x^{k} \text{ is given by}$$

$$X^{i}_{,jk}=\partial_k (X^{i}_{,j})+\dot{\partial}_l (X^{i}_{,j})\, \partial_k X^{l}+X^{l}_{,j} \overset{1}{\Gamma}{}^{i}_{lk}-X^{i}_{,l}\overset{1}{\Gamma}{}^{l}_{jk}$$

$$=\partial_k \partial_j X^{i}+X^{l} \partial_k \overset{1}{\Gamma}{}^{i}_{lj}+\overset{1}{\Gamma}{}^{i}_{lj} \partial_k X^{l}+X^{l} \dot{\partial}_h \overset{1}{\Gamma}{}^{i}_{lj} \partial_k X^{h}$$

$$+\partial_j X^{l} \overset{1}{\Gamma}{}^{i}_{lk}+X^{l} \overset{1}{\Gamma}{}^{h}_{lj} \overset{1}{\Gamma}{}^{i}_{hk}-X^{i}_{,l} \overset{1}{\Gamma}{}^{l}_{jk} \qquad \ldots \quad (2.1)$$

On interchanging j and k in the equation (2.1), and subtracting the equation thus obtained, from (2.1), we have

$$X^{i}_{,jk}-X^{i}_{,kj}=\overset{1}{\tilde{K}}{}^{i}_{ljk}X^{l}-X^{i}_{,l}(\overset{1}{\Gamma}{}^{l}_{jk}-\overset{1}{\Gamma}{}^{l}_{kj}),$$

where $\overset{1}{\tilde{K}}{}^{i}_{ljk}$ is a tensor of the type (1, 3) defined by

$$\overset{1}{\tilde{K}}{}^{i}_{ljk}= \mathbf{G}_{jk}\{\dot{\partial}_k\overset{1}{\Gamma}{}^{i}_{lj}+\dot{\partial}_h\overset{1}{\Gamma}{}^{i}_{lj}\partial_k X^{h}+\overset{1}{\Gamma}{}^{h}_{lj}\overset{1}{\Gamma}{}^{i}_{hk}\}. \qquad \ldots \quad (2.2)$$

On substituting $\partial_k X^{h}=-\overset{1}{\Gamma}{}^{h}_{k}$ in (2.2), the tensor (2.2) becomes

$$\overset{1}{\tilde{K}}{}^{i}_{ljk}= \mathbf{G}_{jk}\{\partial_k\overset{1}{\Gamma}{}^{i}_{lj}-\dot{\partial}_h\overset{1}{\Gamma}{}^{i}_{lj}\overset{1}{\Gamma}{}^{h}_{k}+\overset{1}{\Gamma}{}^{h}_{lj}\overset{1}{\Gamma}{}^{i}_{hk}\}. \qquad \ldots \quad (2.3)$$

Using the definition (1.5) and proceeding similarly as above, we can show that

$$Y_{i,jk}-Y_{i,kj}=-\overset{2}{\tilde{K}}{}^{l}_{ijk}Y_l-Y_{i,l}(\overset{2}{\Gamma}{}^{l}_{jk}-\overset{2}{\Gamma}{}^{l}_{kj}),$$

where $\overset{2}{\tilde{K}}{}^{l}_{ijk}$ is a tensor of the type (1, 3) defined by

$$\overset{2}{\tilde{K}}{}^{l}_{ijk}= \mathbf{G}_{jk}\{\partial_k\overset{2}{\Gamma}{}^{l}_{ij}+\dot{\partial}^{h}\overset{2}{\Gamma}{}^{l}_{ij}\partial_k Y_h+\overset{2}{\Gamma}{}^{l}_{hk}\overset{2}{\Gamma}{}^{h}_{ij}\}. \qquad \ldots \quad (2.4)$$

On substituting $\partial_k Y_h=\overset{2}{\Gamma}_{hk}$ in (2.4), the tensor (2.4) becomes

$$\overset{2}{K}{}^{l}_{ijk}=\mathbf{G}_{jk}\{\partial_k\overset{2}{\Gamma}{}^{l}_{ij}+\dot{\partial}^{h}\overset{2}{\Gamma}{}^{l}_{ij}\overset{2}{\Gamma}_{hk}+\overset{2}{\Gamma}{}^{l}_{hk}\overset{2}{\Gamma}{}^{h}_{ij}\}. \qquad \ldots \quad (2.5)$$

Thus we have a set of four curvature tensors $\overset{1}{\tilde{K}}{}^{i}_{ljk}$, $\overset{1}{K}{}^{i}_{ljk}$, $\overset{2}{\tilde{K}}{}^{l}_{ijk}$ and $\overset{2}{K}{}^{l}_{ijk}$ defined with the help of non-linear connections. From (2.2), (2.3), (2.4), (2.5), (1.4) and (1.5), we have

$$\overset{1}{\tilde{K}}{}^{i}_{ljk}-\overset{1}{K}{}^{i}_{ljk}=\mathbf{G}_{jk}\{\dot{\partial}_h\overset{1}{\Gamma}{}^{i}_{lj}X^{h}_{,k}\},\ \overset{2}{\tilde{K}}{}^{l}_{ijk}-\overset{2}{K}{}^{l}_{ijk}=\mathbf{G}_{jk}\{\dot{\partial}^{h}\overset{2}{\Gamma}{}^{l}_{ij}Y_{h,k}\},$$

$$\overset{1}{K}{}^{i}_{ljk}\, X^{l}= G_{jk}\{\partial_k \overset{1}{\Gamma}{}^{i}_{j} - \overset{1}{\Gamma}{}^{i}_{hj}\overset{1}{\Gamma}{}^{h}_{k}\}, \qquad \dots \quad (2.6)$$

$$\overset{2}{K}{}^{l}_{ijk}\, Y_{l}= G_{jk}\{\partial_k \overset{2}{\Gamma}{}_{ij} + \overset{2}{\Gamma}{}^{h}_{ij}\overset{2}{\Gamma}{}_{hk}\}. \qquad \dots \quad (2.7)$$

3. Commutation formulae for the tensor field T^i_j depending on $\overset{1}{\Gamma}{}^{i}_{j}$: In this article, we will obtain the commutation formulae for a tensor field T^i_j. According to definition (1.4), the covariant derivative of $T^i_{j,k}$ w. r. t. x^l is given by

$$\begin{aligned}
T^i_{j,kl} &= \partial_l T^i_{j,k} + \dot{\partial}_h T^i_{j,k}\partial_l X^h + T^h_{j,k}\overset{1}{\Gamma}{}^{i}_{hl} - T^i_{h,k}\overset{1}{\Gamma}{}^{h}_{jl} - T^i_{j,h}\overset{1}{\Gamma}{}^{h}_{kl} \\
&= \partial_l \dot{\partial}_r T^i_j \partial_k X^r + \partial_l T^r_j\, \overset{1}{\Gamma}{}^{i}_{rk} + T^r_j \partial_l \overset{1}{\Gamma}{}^{i}_{rk} - \partial_l T^i_r \overset{1}{\Gamma}{}^{r}_{jk} - T^i_r \partial_l \overset{1}{\Gamma}{}^{r}_{jk} + (\dot{\partial}_h \partial_k T^i_j \\
&\quad + \dot{\partial}_h T^r_j\, \overset{1}{\Gamma}{}^{i}_{rk} + T^r_j \dot{\partial}_h \overset{1}{\Gamma}{}^{i}_{rk} - \dot{\partial}_h T^i_r \overset{1}{\Gamma}{}^{r}_{jk} - T^i_r \dot{\partial}_h \overset{1}{\Gamma}{}^{r}_{jk})\partial_l X^h + (\partial_k T^h_j \\
&\quad + \dot{\partial}_r T^h_j \partial_k X^r + T^r_j \overset{1}{\Gamma}{}^{h}_{rk} - T^h_r \overset{1}{\Gamma}{}^{r}_{jk})\, \overset{1}{\Gamma}{}^{i}_{hl} - (\partial_k T^i_h + \dot{\partial}_r T^i_h \partial_k X^r + T^r_h \overset{1}{\Gamma}{}^{i}_{rk} \\
&\quad - T^i_r \overset{1}{\Gamma}{}^{r}_{hk})\, \overset{1}{\Gamma}{}^{h}_{jl} - T^i_{j,h}\overset{1}{\Gamma}{}^{h}_{kl} + \text{terms symmetric in } k,\, l. \qquad \dots \quad (3.1)
\end{aligned}$$

On interchanging k and l in the equation (3.1), and subtracting the equation thus obtained, from (3.1) and substituting (2.2), we have

$$T^i_{j,kl} - T^i_{j,lk} = T^r_j \overset{1}{\tilde{K}}{}^{i}_{rkl} - T^i_r\, \overset{1}{\tilde{K}}{}^{r}_{jkl} - T^i_{j,h}(\overset{1}{\Gamma}{}^{h}_{kl} - \overset{1}{\Gamma}{}^{h}_{lk}).$$

We can similarly show that

$$T^{i_1\dots i_r}_{j_1\dots j_s,kl} \quad - T^{i_1\dots i_r}_{j_1\dots j_s,lk}$$

$$= \sum_{\mu=1}^{r} T^{i_1\dots i_{\mu-1} p\, i_{\mu+1}\dots i_r}_{j_1\dots j_s} \overset{1}{\tilde{K}}{}^{i_\mu}_{pkl} - \sum_{\nu=1}^{s} T^{i_1\dots i_r}_{j_1\dots j_{\nu-1} p\, j_{\nu+1}\dots j_s}\, \overset{1}{\tilde{K}}{}^{p}_{j_\nu kl} - T^{i_1\dots i_r}_{j_1\dots j_s,h}\,(\overset{1}{\Gamma}{}^{h}_{kl} - \overset{1}{\Gamma}{}^{h}_{lk}).$$

Now according to definition (1.6), the covariant derivative of $T^i_{j|k}$ w. r. t. x^l is given by

$$\begin{aligned}
T^i_{j|kl} &= \partial_l T^i_{j|k} - \dot{\partial}_h T^i_{j|k}\, \overset{1}{\Gamma}{}^{h}_{l} + T^h_{j|k}\, \overset{1}{\Gamma}{}^{i}_{hl} - T^i_{h|k}\, \overset{1}{\Gamma}{}^{h}_{jl} - T^i_{j|h}\, \overset{1}{\Gamma}{}^{h}_{kl} \\
&= -\partial_l \dot{\partial}_r\, T^i_j \overset{1}{\Gamma}{}^{r}_{k} - \dot{\partial}_r T^i_j \partial_l \overset{1}{\Gamma}{}^{r}_{k} + \partial_l T^r_j \overset{1}{\Gamma}{}^{i}_{rk} + T^r_j \partial_l \overset{1}{\Gamma}{}^{i}_{rk} - \partial_l T^i_r \overset{1}{\Gamma}{}^{r}_{jk} - T^i_r \partial_l \overset{1}{\Gamma}{}^{r}_{jk} - (\dot{\partial}_h \partial_k T^i_j \\
&\quad - \dot{\partial}_r T^i_j \overset{1}{\Gamma}{}^{r}_{hk} + \dot{\partial}_h T^r_j \overset{1}{\Gamma}{}^{i}_{rk} + T^r_j \dot{\partial}_h \overset{1}{\Gamma}{}^{i}_{rk} - \dot{\partial}_h T^i_r \overset{1}{\Gamma}{}^{r}_{jk} - T^i_r \dot{\partial}_h \overset{1}{\Gamma}{}^{r}_{jk})\, \overset{1}{\Gamma}{}^{h}_{l} + (\partial_k T^h_j - \dot{\partial}_r T^h_j\, \overset{1}{\Gamma}{}^{r}_{h} \\
&\quad + T^r_j \overset{1}{\Gamma}{}^{h}_{rk} - T^h_r \overset{1}{\Gamma}{}^{r}_{jk})\, \overset{1}{\Gamma}{}^{i}_{hl} - (\partial_k T^i_h - \dot{\partial}_r T^i_h \overset{1}{\Gamma}{}^{r}_{k} + T^r_h \overset{1}{\Gamma}{}^{i}_{rk} - T^i_r \overset{1}{\Gamma}{}^{i}_{hk})\, \overset{1}{\Gamma}{}^{h}_{jl} - T^i_{j|h} \overset{1}{\Gamma}{}^{h}_{kl} \\
&\quad + \text{terms symmetric in } k,\, l. \qquad \dots \quad (3.2)
\end{aligned}$$

On interchanging k and l in the equation (3.2), and subtracting the equation thus obtained, from (3.2) and substituting (2.6) and (2.3), we have

$$T^i_{j|kl}-T^i_{j|lk}=-\dot{\partial}_r T^i_j \overset{1}{K}{}^r_{hkl} X^h+T^h_j \overset{1}{K}{}^i_{hkl}-T^i_h \overset{1}{K}{}^h_{jkl} T^i_{j|h}(\overset{1}{\Gamma}{}^h_{kl}-\overset{1}{\Gamma}{}^h_{lk}).$$

Similarly we can show that

$$T^{i_1 \dots i_r}_{j_1 \dots j_s|kl}-T^{i_1 \dots i_r}_{j_1 \dots j_s|lk}=-\dot{\partial}_r T^{i_1 \dots i_r}_{j_1 \dots j_s} \overset{1}{K}{}^r_{hkl} X^h+\sum_{\mu=1}^{r} T^{i_1 \dots i_{\mu-1} p\, i_{\mu+1} \dots i_r}_{j_1 \dots j_s} \overset{1}{K}{}^{i_\mu}_{pkl}$$

$$-\sum_{\nu=1}^{s} T^{i_1 \dots i_r}_{j_1 \dots j_{\nu-1} p\, j_{\nu+1} \dots j_s} \overset{1}{K}{}^p_{j_\nu kl}-T^{i_1 \dots i_r}_{j_1 \dots j_s|h}(\overset{1}{\Gamma}{}^h_{kl}-\overset{1}{\Gamma}{}^h_{lk}).$$

4. Commutation formulae for the tensor field $\bar{T}^i_j$ depending on $\overset{2}{\Gamma}_{jk}$: In this article, we will obtain the commutation formulae for a tensor field $\bar{T}^i_j$. According to definition (1.5), the covariant derivative of $\bar{T}^i_{j;k}$ w. r. t. x^l is given by

$$\bar{T}^i_{j;kl}=\partial_l \bar{T}^i_{j;k}+\dot{\partial}^h \bar{T}^i_{j;k} \partial_l Y_h+\bar{T}^h_{j;k} \overset{2}{\Gamma}{}^i_{hl}-\bar{T}^i_{h;k} \overset{2}{\Gamma}{}^h_{jl}-\bar{T}^i_{j;h} \overset{2}{\Gamma}{}^h_{kl}$$

$$=\partial_l \dot{\partial}^r \bar{T}^i_j \partial_k Y_r+\partial_l \bar{T}^r_j \overset{2}{\Gamma}{}^i_{rk}+\bar{T}^r_j \partial_l \overset{2}{\Gamma}{}^i_{rk}-\partial_l \bar{T}^i_r \overset{2}{\Gamma}{}^r_{jk}-\bar{T}^i_r \partial_l \overset{2}{\Gamma}{}^r_{jk}+(\dot{\partial}^h \partial_k \bar{T}^i_j$$

$$+\dot{\partial}^h \bar{T}^r_j \overset{2}{\Gamma}{}^i_{rk}+\bar{T}^r_j \dot{\partial}^h \overset{2}{\Gamma}{}^i_{rk}-\dot{\partial}^h \bar{T}^i_r \overset{2}{\Gamma}{}^r_{jk}-\bar{T}^i_r \dot{\partial}^h \overset{2}{\Gamma}{}^r_{jk}) \partial_l Y_h+(\dot{\partial}_k \bar{T}^h_j+\dot{\partial}^r \bar{T}^h_j \partial_k Y_r$$

$$+\bar{T}^r_j \overset{2}{\Gamma}{}^h_{rk}-\bar{T}^h_r \overset{2}{\Gamma}{}^r_{jk}) \overset{2}{\Gamma}{}^i_{hl}-(\partial_k \bar{T}^i_h+\dot{\partial}^r \bar{T}^i_h \partial_k Y_r+\bar{T}^r_h \overset{2}{\Gamma}{}^i_{rk}-\bar{T}^i_r \overset{2}{\Gamma}{}^r_{hk}) \overset{2}{\Gamma}{}^h_{jl}$$

$$-\bar{T}^i_{j;h} \overset{2}{\Gamma}{}^h_{kl}+\text{terms symmetric in } k, l. \qquad \dots \quad (4.1)$$

On interchanging k and l in the equation (4.1), and subtracting the equation thus obtained, from (4.1) and substituting (2.4), we have

$$\bar{T}^i_{j;kl}-\bar{T}^i_{j;lk}=\bar{T}^r_j \overset{2}{\tilde{K}}{}^i_{rkl}-\bar{T}^i_r \overset{2}{\tilde{K}}{}^r_{jkl}-\bar{T}^i_{j;h}(\overset{2}{\Gamma}{}^h_{kl}-\overset{2}{\Gamma}{}^h_{lk}).$$

Similarly we can show that

$$\bar{T}^{i_1 \dots i_r}_{j_1 \dots j_s;kl}-\bar{T}^{i_1 \dots i_r}_{j_1 \dots j_s;lk}$$

$$=\sum_{\mu=1}^{r} \bar{T}^{i_1 \dots i_{\mu-1} p\, i_{\mu+1} \dots i_r}_{j_1 \dots j_s} \overset{2}{\tilde{K}}{}^{i_\mu}_{pkl}-\sum_{\nu=1}^{s} \bar{T}^{i_1 \dots i_r}_{j_1 \dots j_{\nu-1} p\, j_{\nu+1} \dots j_s} \overset{2}{\tilde{K}}{}^p_{j_\nu kl}$$

$$-\bar{T}^{i_1 \dots i_r}_{j_1 \dots j_s; h}(\overset{2}{\Gamma}{}^h_{kl}-\overset{2}{\Gamma}{}^h_{lk})$$

Now according to definition (1.7), the covariant derivative of $\bar{T}^i_{j|k}$ w. r. t. x^l is given by

$$\bar T^i_{j|kl} = \partial_l \bar T^i_{j|k} + \dot\partial^h \bar T^i_{j|k}\, \overset{2}{\Gamma}_{hl} + \bar T^h_{j|k}\, \overset{2}{\Gamma}{}^i_{hl} - \bar T^i_{h|k}\, \overset{2}{\Gamma}{}^h_{jl} - \bar T^i_{j|h}\, \overset{2}{\Gamma}{}^h_{kl}$$

$$= \partial_l \dot\partial^r \bar T^i_j\, \overset{2}{\Gamma}_{rk} + \dot\partial^r \bar T^i_j\, \partial_l \overset{2}{\Gamma}_{rk} + \partial_l \bar T^r_j\, \overset{2}{\Gamma}{}^i_{rk} + \bar T^r_j\, \partial_l \overset{2}{\Gamma}{}^i_{rk} - \partial_l \bar T^i_r\, \overset{2}{\Gamma}{}^r_{jk} - \bar T^i_r\, \partial_l \overset{2}{\Gamma}{}^r_{jk}$$

$$+ (\dot\partial^h \partial_k \bar T^i_j + \dot\partial^r \bar T^i_j\, \overset{2}{\Gamma}{}^h_{rk} + \dot\partial^h \bar T^r_j\, \overset{2}{\Gamma}{}^i_{rk} + \bar T^r_j\, \dot\partial^h \overset{2}{\Gamma}{}^i_{rk} - \dot\partial^h \bar T^i_r\, \overset{2}{\Gamma}{}^r_{jk} - T^i_r \dot\partial^h \overset{2}{\Gamma}{}^r_{jh})\, \overset{2}{\Gamma}_{hl}$$

$$+ (\partial_k \bar T^h_j + \dot\partial^r \bar T^h_j\, \overset{2}{\Gamma}_{rk} + \bar T^r_j\, \overset{2}{\Gamma}{}^h_{rk} - \bar T^h_r\, \overset{2}{\Gamma}{}^r_{jk})\, \overset{2}{\Gamma}{}^i_{hl} - (\partial_k \bar T^i_h + \dot\partial^r \bar T^i_h\, \overset{2}{\Gamma}_{rk} + \bar T^r_h\, \overset{2}{\Gamma}{}^i_{rk}$$

$$- \bar T^i_r\, \overset{2}{\Gamma}{}^r_{hk})\, \overset{2}{\Gamma}{}^h_{jl} - \bar T^i_{j|h}\, \overset{2}{\Gamma}{}^h_{kl} + \text{terms symmetric in } k, l \qquad \dots \quad (4.2)$$

On interchanging k and l in the equation (4.2), and subtracting the equation thus obtained, from (4.2) and substituting (2.7) and (2.5), we have

$$\bar T^i_{j|kl} - \bar T^i_{j|lk} = \dot\partial^r \bar T^i_j\, \overset{2}{K}{}^h_{rkl}\, Y_h + \bar T^r_j\, \overset{2}{K}{}^i_{rkl} - \bar T^i_r\, \overset{2}{K}{}^r_{jkl} - \bar T^i_{j|h}\, (\overset{2}{\Gamma}{}^h_{kl} - \overset{2}{\Gamma}{}^h_{lk}$$

Similarly we can show that

$$\bar T^{i_1 \dots i_r}_{j_1 \dots j_s|kl} - \bar T^{i_1 \dots i_r}_{j_1 \dots j_s|lk} = \dot\partial^r \bar T^{i_1 \dots i_r}_{j_1 \dots j_s}\, \overset{2}{K}{}^h_{rkl}\, Y_h + \sum_{\mu=1}^{r} \bar T^{i_1 \dots i_{\mu-1} p i_{\mu+1} \dots i_r}_{j_1 \dots j_s}\, \overset{2}{K}{}^{i_\mu}_{pkl}$$

$$- \sum_{\nu=1}^{s} \bar T^{i_1 \dots i_r}_{j_1 \dots j_{\nu-1} p j_{\nu+1} \dots j_s}\, \overset{2}{K}{}^p_{j_\nu kl} - \bar T^{i_1 \dots i_r}_{j_1 \dots j_s|h}\, (\overset{2}{\Gamma}{}^h_{kl} - \overset{2}{\Gamma}{}^h_{lk}).$$

5. Covariant derivatives depending on $\overset{1}{\Gamma}{}^i_j$ and $\overset{2}{\Gamma}_{jk}$ both and commutation formulae

Let X^i and Y_i be related by $Y_i = g_{ij}(x, X)\, X^j$. Let us put

$$\overset{1}{\bar\Gamma}{}^i_{jk}(x, Y) = \overset{1}{\Gamma}{}^i_{jk}(x, X(x, Y)),\ \overset{2}{\bar\Gamma}{}^i_{jk}(x, X) = \overset{2}{\Gamma}{}^i_{jk}(x, Y(x, X)) \qquad \dots \quad (5.1)$$

Let two curvature tensors $\overset{1}{\tilde K}{}^i_{ljk}$ and $\overset{2}{\tilde K}{}^l_{ijk}$ be defined by

$$\overset{1}{\tilde K}{}^i_{ljk} = \mathbf{G}_{jk}\, \{\partial_k \overset{1}{\bar\Gamma}{}^i_{lj} + \dot\partial^h \overset{1}{\bar\Gamma}{}^i_{lj}\, \overset{2}{\bar\Gamma}_{hk} + \overset{1}{\bar\Gamma}{}^h_{lj}\, \overset{1}{\bar\Gamma}{}^i_{hk}\}, \qquad \dots \quad (5.2)$$

$$\overset{2}{\tilde K}{}^l_{ijk} = \mathbf{G}_{jk}\, \{\partial_k \overset{2}{\bar\Gamma}{}^l_{ij} - \dot\partial_h \overset{2}{\bar\Gamma}{}^l_{ij}\, \overset{1}{\Gamma}{}^h_k + \overset{2}{\bar\Gamma}{}^l_{hk}\, \overset{2}{\bar\Gamma}{}^h_{ij}\}, \qquad \dots \quad (5.3)$$

From (2 2), (2.4) and (5.1), we have

$$\overset{1}{\tilde K}{}^i_{ljk} = \mathbf{G}_{jk}\, \{\partial_k \overset{1}{\bar\Gamma}{}^i_{lj} + \partial^h \overset{1}{\bar\Gamma}{}^i_{lj}\, \partial_k Y_h + \overset{1}{\bar\Gamma}{}^h_{lj}\, \overset{1}{\bar\Gamma}{}^i_{hk}\}, \qquad \dots \quad (5.4)$$

$$\overset{2}{\tilde K}{}^l_{ijk} = \mathbf{G}_{jk}\, \{\partial_k \overset{2}{\bar\Gamma}{}^l_{ij} + \dot\partial_h \overset{2}{\bar\Gamma}{}^l_{ij}\, \partial_k X^h + \overset{2}{\bar\Gamma}{}^l_{hk}\, \overset{2}{\bar\Gamma}{}^h_{ij}\}. \qquad \dots \quad (5.5)$$

By virtue of (1.4), (1.5), (5.2), (5.3) (5.4) and (5.5), we obtain

$$\overset{1}{\tilde K}{}^i_{ljk}-\overset{1}{\bar K}{}^i_{ljk}=\mathbf{G}_{jk}\,\{\dot\partial^h\,\overset{1}{\bar\Gamma}{}^i_{lj}\,Y_{h;k}\},\ \overset{2}{\tilde K}{}^l_{ijk}-\overset{2}{\tilde K}{}^l_{ijk}=\mathbf{G}_{jk}\,\{\dot\partial_h\,\overset{2}{\bar\Gamma}{}^l_{ij}\,X^h_{,k}\}.$$

We may define covariant derivatives in several other ways with the help of non-linear connection. But we shall consider only two of them which depend on $\overset{1}{\Gamma}{}^i_j$ and $\overset{2}{\Gamma}_{jk}$ both. Following Kikuchi (1962), we can define two types of covariant derivatives for arbitrary tensors $\bar T^i_j$ and T^i_j (say), which are

$$\bar T^i_{j\Delta k}=\partial_k\,\bar T^i_j+\dot\partial^h\,\bar T^i_j\,\overset{2}{\Gamma}_{hk}+\bar T^h_j\,\overset{1}{\bar\Gamma}{}^i_{hk}-\bar T^i_h\,\overset{1}{\bar\Gamma}{}^h_{jk}, \qquad \ldots (5.6)$$

$$\bar T^i_{j\nabla k}=\partial_k\,\bar T^i_j-\dot\partial_h\,\bar T^i_j\,\overset{1}{\Gamma}{}_k{}^h+\bar T^h_j\,\overset{2}{\bar\Gamma}{}^i_{hk}-\bar T^i_h\,\overset{2}{\bar\Gamma}{}^h_{jk}. \qquad \ldots (5.7)$$

According to definition (5.6), the covariant derivative of $\bar T^i_{j\Delta k}$ w. r. t. x^l is given by

$$\begin{aligned}\bar T^i_{j\Delta kl}=&\partial_l\,\bar T^i_{j\Delta k}+\dot\partial^h\,\bar T^i_{j\Delta k}\,\overset{2}{\Gamma}_{hl}+\bar T^i_{j\Delta k}\,\overset{1}{\bar\Gamma}{}^h_{hl}-\bar T^i_{h\Delta k}\,\overset{1}{\bar\Gamma}{}^h_{jl}-\bar T^i_{j\Delta h}\,\overset{1}{\bar\Gamma}{}^h_{kl}\\
=&\partial_l\,\dot\partial^r\,\bar T^i_j\,\overset{2}{\Gamma}_{rk}+\dot\partial^r\,\bar T^i_j\,\partial_l\,\overset{2}{\Gamma}_{rk}+\partial_l\,\bar T^r_j\,\overset{1}{\bar\Gamma}{}^i_{rk}+\bar T^r_j\,\partial_l\,\overset{1}{\bar\Gamma}{}^i_{rk}-\partial_l\,\bar T^i_r\,\overset{1}{\bar\Gamma}{}^r_{jk}-\bar T^i_r\,\partial_l\,\overset{1}{\bar\Gamma}{}^r_{jk}\\
&+(\dot\partial^h\,\partial_k\,\bar T^i_j+\dot\partial^r\,\bar T^i_j\,\overset{2}{\Gamma}{}_{rk}^h+\dot\partial^h\,\bar T^r_j\,\overset{1}{\bar\Gamma}{}^i_{rk}+\bar T^r_j\,\dot\partial^h\,\overset{1}{\bar\Gamma}{}^i_{rk}-\dot\partial^h\,\bar T^i_r\,\overset{1}{\bar\Gamma}{}^r_{jk}-\bar T^i_r\,\dot\partial^h\,\overset{1}{\bar\Gamma}{}^r_{jk})\,\overset{2}{\Gamma}_{hl}\\
&+(\partial_k\,\bar T^h_j+\dot\partial^r\,\bar T^h_j\,\overset{2}{\Gamma}_{rk}+\bar T^r_j\,\overset{1}{\bar\Gamma}{}^h_{rk}-\bar T^h_r\,\overset{1}{\bar\Gamma}{}^r_{jk})\,\overset{1}{\bar\Gamma}{}^i_{hl}-(\partial_k\,\bar T^i_h+\dot\partial^r\,\bar T^i_h\,\overset{2}{\Gamma}_{rk}+\bar T^r_h\,\overset{1}{\bar\Gamma}{}^i_{rk}\\
&-\bar T^i_r\,\overset{1}{\bar\Gamma}{}^r_{hk})\,\overset{1}{\bar\Gamma}{}^h_{jb}-T^i_{j\Delta h}\,\overset{1}{\bar\Gamma}{}^h_{kl}+\text{terms symmetric in } k,\ l.\end{aligned} \qquad \ldots (5.8)$$

On interchanging k and l in the equation (5.8), and subtracting the equation thus obtained from (5.8) and substituting (2.7) and (5.2), we have

$$\bar T^i_{j\Delta kl}-\bar T^i_{j\Delta kl}=\dot\partial^r\,\bar T^i_j\,\overset{2}{K}{}^h_{rkl}\,Y_h+\bar T^r_j\,\overset{1}{\bar K}{}^i_{rkl}-\bar T^i_r\,\overset{1}{\bar K}{}^r_{jkl}-\bar T^i_{j\Delta h}\,(\overset{1}{\bar\Gamma}{}^h_{kl}-\overset{1}{\bar\Gamma}{}^h_{lk}).$$

Similarly we can show that

$$\begin{aligned}\bar T^{i_1\cdots i_r}_{j_1\cdots j_s\Delta kl}-\bar T^{i_1\cdots i_r}_{j_1\cdots j_s\Delta lk}\,\bar T=&\dot\partial^r\,\overset{2}{K}{}^h_{rkl}\,Y_h+\sum_{\mu=1}^{r}\bar T^{i_1\cdots i_{\mu-1}pi_{\mu+1}\cdots i_r}_{j_1\cdots j_s}\,\overset{1}{\bar K}{}^{i_\mu}_{pkl}\\
&-\sum_{v=1}^{s}\bar T^{i_1\cdots i_r}_{j_1\cdots j_{v-1}pj_{v+1}\cdots j_s}\,\overset{1}{\bar K}{}^p_{j_vkl}-\bar T^{i_1\cdots i_r}_{j_1\cdots j_s\Delta h}\,(\overset{1}{\bar\Gamma}{}^h_{kl}-\overset{1}{\bar\Gamma}{}^h_{lk}).\end{aligned} \qquad \ldots$$

Now according to definition (5.7), the covariant derivative of $T^i_{j\nabla k}$ w. r. t. x^l is given by

$$T^i_{j\nabla kl}=\partial_l\,T^i_{j\nabla k}-\dot\partial_h\,T^i_{j\nabla k}\,\overset{1}{\bar\Gamma}{}_l{}^h+T^h_{j\nabla k}\,\overset{2}{\bar\Gamma}{}^i_{hl}-T^i_{h\nabla k}\,\overset{2}{\bar\Gamma}{}^h_{jl}-T^i_{j\nabla h}\,\overset{2}{\bar\Gamma}{}^h_{kl}$$

$$=-\partial_l\,\dot{\partial}_r T^i_j \overset{1}{\Gamma}{}^r_k-\dot{\partial}_r T^i_j\,\partial_l \overset{1}{\Gamma}{}^r_k+\partial_l T^r_j \overset{2}{\bar{\bar{\Gamma}}}{}^i_{rk}+T^r_j\,\partial_l \overset{2}{\bar{\bar{\Gamma}}}{}^i_{rk}-\partial_l T^i_r \overset{2}{\bar{\bar{\Gamma}}}{}^r_{jk}-T^i_r\,\partial_l \overset{2}{\bar{\bar{\Gamma}}}{}^r_{jk}$$

$$-(\dot{\partial}_h\,\partial_k T^i_j-\dot{\partial}_r T^i_j \overset{1}{\Gamma}{}^r_{hk}+\dot{\partial}_h T^r_j \overset{2}{\bar{\bar{\Gamma}}}{}^i_{rk}+T^r_j\;\dot{\partial}_h \overset{2}{\bar{\bar{\Gamma}}}{}^i_{rk}-\dot{\partial}_h T^i_r \overset{2}{\bar{\bar{\Gamma}}}{}^r_{jk}-T^i_r\;\dot{\partial}_h \overset{2}{\bar{\bar{\Gamma}}}{}^r_{jk})\overset{1}{\Gamma}{}^h_l$$

$$+(\partial_k T^h_j-\dot{\partial}_r T^h_j \overset{1}{\Gamma}{}^r_k+T^r_j \overset{2}{\bar{\bar{\Gamma}}}{}^h_{rk}-T^h_r \overset{2}{\bar{\bar{\Gamma}}}{}^r_{jk})\overset{2}{\bar{\bar{\Gamma}}}{}^i_{hl}-(\partial_k T^i_h-\dot{\partial}_r T^i_h \overset{1}{\Gamma}{}^r_k+T^r_h \overset{2}{\bar{\bar{\Gamma}}}{}^i_{rk}$$

$$-T^i_r \overset{2}{\bar{\bar{\Gamma}}}{}^r_{hk})\overset{2}{\bar{\bar{\Gamma}}}{}^h_{jl}-T^i_{j\nabla h} \overset{2}{\bar{\bar{\Gamma}}}{}^h_{kl}+\text{terms symmetric in } k,\, l \qquad \ldots \quad (5.9)$$

On interchanging k and 1 in the equation (5.9), and subtracting the equation thus obtained, from (5.9) and substituting (2.6) and (5.3), we have

$$T^i_{j\nabla kl}-T^i_{j\nabla lk}=-\dot{\partial}_r T^i_j \overset{1}{K}{}^r_{hkl}\,X^h+T^r_j \overset{2}{\bar{K}}{}^i_{rkl}-T^i_r \overset{2}{\bar{K}}{}^r_{jkl}-T^i_{j\nabla h}(\overset{2}{\bar{\bar{\Gamma}}}{}^h_{kl}-\overset{2}{\bar{\bar{\Gamma}}}{}^h_{lk}).$$

Similarly we can show that

$$T^{i_1\cdots i_r}_{j_1\cdots j_s\nabla kl}-T^{i_1\cdots i_r}_{j_1\cdots j_s\nabla lk}=-\dot{\partial}_r T^{i_1\cdots i_r}_{j_1\cdots j_s} \overset{1}{K}{}^r_{hkl}\,X^h \sum_{\mu=1}^{r} T^{i_1\cdots i_\mu\,1 p\, i_{\mu+1}\cdots i_r}_{j_1\cdots j_s} \overset{2}{\bar{K}}{}^{i_\mu}_{p\,l}$$

$$-\sum_{v=1}^{s} T^{i_1\cdots i_r}_{j_1\cdots j_{v-1}\,p\,j_{v+1}\cdots j_s} \overset{2}{\bar{K}}{}^p_{j_v kl}-T^{i_1\cdots i_r}_{j_1\cdots j_s\nabla h}(\overset{2}{\bar{\bar{\Gamma}}}{}^h_{kl}-\overset{2}{\bar{\bar{\Gamma}}}{}^h_{lk}).$$

References

Kawaguchi, A. (1952) : On the theory of non-linear connections I. Introduction to the theory of general non-linear connections. *Tensor* (*N. S*), **2**, 123.

Kawaguchi, A. (1956) : On the theory of non-linear connections II. Theory of Minkowski space and of non-linear connections in a Finsler space. *Tensor* (*N.S*) **6**, 165.

Kikuchi, S. (1962) : Theory of Minkowski space and of non-linear connection in a Finsler space. *Tensor* (*N. S*), **12**, **1**, 47.

Rund, H. (1959) : The differential geometry of Finsler spaces. *Springer-Verlag, Berlin. Gottingen*. Heidelberg.

DEPTT. OF MATHS., UNIVERSITY OF GORAKHPUR, GORAKHPUR.
and
DEPTT. OF MATHS., P. G. COLLEGE, GHAZIPUR.

Bull Cal. Math. Soc., **75**, 303-316 (1983)

FLOW FIELD ANALYSIS ON THE BOOSTER DURING TANDEM STAGE SEPARATION

K. SANKARA RAO and J. PANDURANGAN

(*Received 2 November 1981*)

Abstract : Flow field on the surface of the booster due to normal impingement of non-uniform axisymmetric underexpanded supersonic jet of the upper stage, during stage separation of a multi-stage rocket has been analysed numerically. The pressure distribution, Mach number distribution on the booster surface kept at various distances from the nozzle, normal to the jet axis have been computed together with shock stand off distance and the shape of the bow shock in each case. In the present investigation using the method of Integral relations, the normal component of the velocity is approximated using both linear and parabolic laws, while the radial component of the velocity is approximated by linear law alone. It is concluded that a single strip approximation, with the assumption that both radial and normal components of the velocities approximated linearly, yields reasonably accurate predictions of the flow field.

1. **Introduction :** Recently, supersonic jet impingement problems have become an increasingly important topic of research because of their practical applications in rocket stage separation during the flight of a multistage launch vehicle, rocket nozzle thrust vector control devices ; fuel injectors for supersonic combustion and so on. In particular, the study of an underexpanded supersonic jet when impinges on the surface of the booster is of practical importance. For example, when an upper stage is ignited soon after the lower stage separation of a multistage rocket, the sustainer jet impinges on the surface of the booster and the knowledge of the pressure field on the booster, the position of the detached shock and the aerodynamic forces acting on the booster surface etc. are vital in understanding the fluid dynamic aspects of stage separation at very high altitudes.

Figure 1 shows the main features of the flow field, and essentially remains the same as described by Sankara Rao *et al.* [1976]. The exhaust gases that comes out of the nozzle, expand rapidly and pass through a bow shock before impinging on the booster surface. When the booster is in the vicinity of the sustainer, it is entirely within the sustainer jet and well within the jet shock diameter.

A formal difficulty arises in an explicit formulation of this problem, because of our prior ignorance of the shock location and its form. Belov *et al.* (1973), Gummer and Hunt (1974) have reported theoretical results in the impingement region, when a supersonic underexpanded jet impinges upon an infinite flat plate. The flow in the impingement region is subsonic. From the studies on subsonic impingement, Belov *et al.* (1973) concluded that the

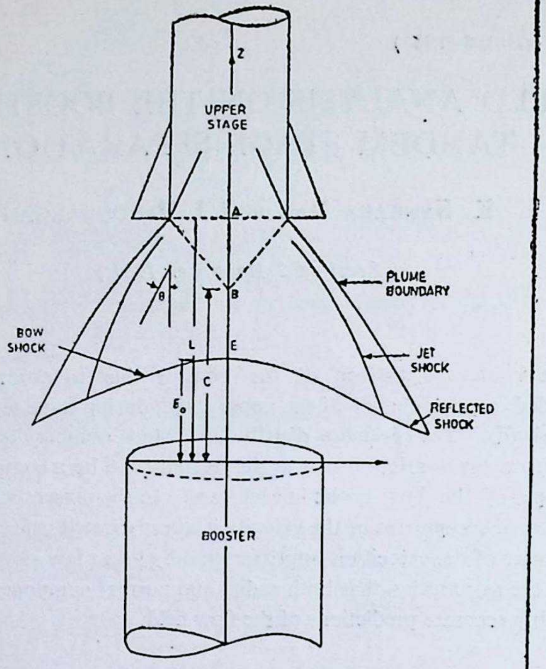


FIG. 1. MODEL OF PLUME IMPINGEMENT DURING TANDOM STAGE SEPARATION

normal velocity varies linearly near the wall but far away follows a parabolic law. However, radial velocity varies only linearly. All the earlier studies [Belov *et al.* 1973, Gummer and Hunt 1974, Sankara Rao *et al.* 1976] are directed with the assumption of linear variation in both normal and radial velocity. In our present investigation we have approximated the normal velocity by both linear and parabolic laws and compared the flow field.

We have considered the problem of computing the flow field on the booster surface due to the impingement of a steady non-uniform axisymmetric underexpanded sustainer jet. A single strip version of the method of integral relations has been used in our present analysis which is due to Belotserkovskii (1958) and is known to yield reasonably accurate results [Xerikos and Anderson 1962].

2. Mathematical Formulation : When a nonuniform supersonic jet impinges on a finite flat surface the main features of the flow field may be considered in two parts, the free jet region which lies above the bow shock produced by the finite flat surface and the impingement region between the bow shock and the plate as shown in Figure 1. In our present investigation we are interested in the impingement region where the flow is subsonic. It is known that the transition from subsonic to supersonic flow occurs in a singular manner at the corner of the booster surface. Further, we assume that the flow in the impingement region is steady, inviscid and adiabatic. The basic governing equations of motion in the cylindrical coordinates after non-dimensionalisation can be written as :

Continuity

$$\frac{\delta}{\delta r}(\rho r u) + \frac{\delta}{\delta z}(\rho r v) = 0 \qquad \dots (1)$$

momentum

$$\frac{\delta}{\delta r}(\rho r u v) + \frac{\delta}{\delta z}\left[r\left(\rho v^2 + \frac{\gamma-1}{2\gamma}\,p\right)\right] = 0 \qquad \dots (2)$$

In addition, the Bernoulli's Integral

$$u^2 + v^2 + p/\rho = 1 \qquad \dots (3)$$

and the equation of state

$$\frac{p}{\rho^{\gamma}} = \phi\,[\chi\,(r, z)] \qquad \dots (4)$$

are required, where ϕ is an entropy function, u and v are velocity components in r and z directions respectively. The pressure p and density ρ are non-dimensionalised with the corresponding stagnation values p_0 and ρ_0 ; the velocity components by the maximum velocity U_{max} that can be attained when the flow exhausts into Vacuum given by $U_{max} = C^* \sqrt{\frac{\gamma+1}{\gamma-1}}$; the spatial coordinates by the nozzle exit radius R_e.

From geometry, the relation between the shock stand off distance $\varepsilon(r)$ and the shock angle σ is given by

$$\frac{d\varepsilon}{dr} = -\cot\sigma \qquad \dots (5)$$

Before we proceed with the interpretation of continuity and momentum equations, the free jet parameters are calculated as follows.

It is assumed that the incoming flow upto the bow shock may be approximated by the flow from the three-dimensional source given by Milne Thomson

$$\eta = [r^2 + (C-\varepsilon)^2]^{\frac{1}{2}} = \sqrt{Q}\left[\frac{q(M_B)}{q(M)} - 1\right]$$

and

$$\theta = \tan^{-1}\left(\frac{r}{L-\varepsilon}\right) \qquad \dots (6)$$

where M_B is the Mach number at the point B [Fig. 1] which is the intersection of leading characteristic emanating from the edge of the nozzle with the axis of symmetry, θ the flow angle, C is the distance from the point B to the plate ; L is the nozzle plate spacing, Q is the intensity of the fictitious source determined from the expression.

$$\sqrt{Q}=\left(\frac{\gamma+1}{\gamma-1}+1\right)^{\frac{1}{2}} / \theta_m \qquad \dots \quad (7)$$

where θ_m is the maximum angle by which the flow undergoes rotation if the expansion takes place in vacuum and

$$q(M)=M\left[\frac{\gamma+1}{2} \Big/ \left(1+\frac{\gamma-1}{2}\ M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

The Mach number M_B can be computed from Prandtl Mayer relation. The Mach number at $E\,(M_E)$ has been computed using the Newtons iteration formula given by

$$M^{(n+1)}=M^{(n)}-\frac{f(M)}{f'(M)}$$

where

$$f(M)=\sqrt{Q}\left\{\frac{q(M_B)}{q(M)}-1\right\}-(c-\varepsilon_0)$$

The distribution of Mach number along the jet axis down stream of the point B is determined on the basis, of calculations by the method of characteristics. The Mach number M may be approximately computed from $M^{(0)}=M_B+\frac{A}{z}\ (\,|\,z-AB\,|\,)^{\gamma}$

where $A=2.35+4.5\ (0.426\ Me-1)\ (0.834\ \gamma-1)$

for $Me > 2$, $\gamma=1.2$ to 1.4, $L > AB$

The parameter downstream of the bow shock can be computed using the following Rankine Hugoniot relations.

$$u_1=\frac{(\gamma-1)^{\frac{1}{2}}}{2}\ M\ a(M)\ [\sin\sigma\cos(\sigma-\theta)-f\cos\sigma\sin(\sigma-\theta)]$$

$$v_1=-\frac{(\gamma-1)^{\frac{1}{2}}}{2}\ M\ a(M)\ [\cos\sigma\cos(\sigma-\theta)+f\sin\sigma\sin(\sigma-\theta)]$$

$$p_1=p\left[\frac{2\gamma}{\gamma+1}\ M^2\sin^2(\sigma-\theta)-\frac{\gamma-1}{\gamma+1}\right]$$

$$\rho_1=\rho\,/\,f$$

$$\phi = \left[\frac{2\gamma}{\gamma+1} M^2 \sin^2(\sigma-\theta) - \frac{\gamma-1}{\gamma+1}\right] f^{\gamma}$$

where

$$f = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M^2 \sin^2(\sigma-\theta)} \quad \text{and}$$

ϕ is the flow entropy,

and

$$\alpha(M) = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{1}{2}}$$

Now considering the entire shock layer as one strip, the continuity and momentum equations (1) and (2) are integrated with respect to Z from the booster surface to the stand off shock. In order to apply the method of Integral relation, an approximation must be made to the integrand, which is assumed to be continuous and bounded. Thus by the Weirstrass theorem, there exists a real polynomial approximation for the integrand. From subsonic jet impingement studies, it is known that the normal velocity varies linearly near the wall but far away follows a parabolic law. However radial velocity nearly follows linear law. Hence, considering parabolic law for normal velocity and linear law for radial velocity, we introduce

$$A = \rho v^2 + \frac{\gamma-1}{2\gamma} p = A_s + (A_1 - A_s)\left[\frac{Z}{\varepsilon}\left(2 - \frac{Z}{\varepsilon}\right)\right]^2$$

$$B = \rho u v = B_1 \left(\frac{Z}{\varepsilon}\right)^2 \left(2 - \frac{Z}{\varepsilon}\right)$$

$$C = \rho u = C_s + \left(\frac{C_1 - C_s}{\varepsilon}\right) Z$$

$$D = \rho v = D_1 \left(\frac{Z}{\varepsilon}\right)\left(2 - \frac{Z}{\varepsilon}\right)$$

(Subscripts S and 1 indicate conditions on the surface of the booster and shock respectively) and arrive at the following non linear coupled ordinary differential equation.

$$\frac{dC_s}{dr} + \frac{dC_1}{dr} + \frac{C_s - C_1}{\varepsilon}\frac{d\varepsilon}{dr} + \frac{C_s + C_1}{r} + \frac{2}{\varepsilon} D_1 = 0 \quad \dots (9)$$

and

$$\frac{dB_1}{dr} + \left(\frac{1}{r} - \frac{7}{5\varepsilon}\frac{d\varepsilon}{dr}\right) B_1 + \frac{12}{5\varepsilon}(A_1 - A_s) = 0 \quad \dots (10)$$

For a non uniform jet, B_1, C_1 and f depends not only on the Mach number M but also on shock angle σ and the flow direction θ. After long algebraic manipulations equations (9) and (10) reduce to

$$P_1 \frac{dM}{dr} + P_2 \frac{d\sigma}{dr} + P_3 \frac{d\theta}{dr} + P_4 = 0 \qquad \dots \quad (11)$$

$$(1-M_s^2) \frac{dM_s}{dr} + Q_1 \frac{dM}{dr} + Q_2 \frac{d\sigma}{dr} + Q_3 \frac{d\theta}{dr} + Q_4 = 0 \qquad \dots \quad (12)$$

where

$$P_1 = \frac{\tau(M)}{2M}(2-M^2)\Big[\frac{\cos^2(\sigma-\theta)\sin 2\sigma}{f} - \cos 2\sigma \sin 2(\sigma-\theta)$$

$$- f\sin^2(\sigma-\theta)\sin 2\sigma\Big] + \frac{2\sin 2\sigma}{(\gamma+1)M^2\sin^2(\sigma-\theta)}\Big\{\frac{\cos^2(\sigma-\theta)}{f^2}$$

$$+ \sin^2(\sigma-\theta)\Big\}$$

$$P_2 = \frac{2\sin 2\sigma \cot(\sigma-\theta)}{(\gamma+1)M^2\sin^2(\sigma-\theta)}\Big\{\frac{\cos^2(\sigma-\theta)}{f^2} + \sin^2(\sigma-\theta)\Big\}$$

$$+ \Big[\frac{\cos(\sigma-\theta)\cos(3\sigma-\theta)}{f} - f\sin(\sigma-\theta)\sin(3\sigma-\theta) - \frac{\cos(\sigma-2\theta)}{1}\Big]$$

$$P_3 = \Big[\frac{\sin 2\sigma \sin 2(\sigma-\theta)}{2}\Big(f+\frac{1}{f}\Big) + \cos 2\sigma \cos 2(\sigma-\theta)$$

$$- \frac{2\sin 2\sigma \cot(\sigma-\theta)}{(\gamma+1)M^2\sin^2(\sigma-\theta)}\Big\{\frac{\cos^2(\sigma-\theta)}{f^2} + \sin^2(\sigma-\theta)\Big\}\Big]$$

and

$$P_4 = \frac{1}{2}\Big(\frac{1}{r} + \frac{7}{5}\,\frac{\cot\sigma}{\varepsilon}\Big)\Big[\frac{\cos^2(\sigma-\theta)\sin 2\sigma}{f} - \frac{\cos 2\sigma \sin 2(\sigma-\theta)}{1}$$

$$- \frac{f\sin^2(\sigma-\theta)\sin 2\sigma}{1}\Big] - \frac{12}{5\varepsilon}\Big[\frac{\cos^2(\sigma-\theta)\cos^2\sigma}{f}$$

$$+ \frac{f\sin^2(\sigma-\theta)\sin^2\sigma}{1} + \frac{\sin 2(\sigma-\theta)\sin 2\sigma}{2}\Big]$$

$$- \frac{12}{5\varepsilon M^2\gamma}\Big[\frac{2\gamma}{\gamma+1}M^2\sin^2(\sigma-\theta) - \frac{\gamma-1}{\gamma+1} - \frac{\pi(M_S)}{\pi(M)}\phi_S^{-1/(\gamma-1)}\Big]$$

$$Q_1 = \frac{\pi(M)\,\alpha(M_S)\,\phi_S^{1/(\gamma-1)}}{\alpha(M)\,\pi(M_S)\,\tau(M_S)}\Big[(1-M^2)\,\tau(M)\Big\{\frac{\cos(\sigma-\theta)}{f}\sin\sigma$$

$$-\frac{\sin(\sigma-\theta)\cos\sigma}{1}\Big\}+\frac{4\cos(\sigma-\theta)\sin\sigma}{f^2(\gamma+1)M^2\sin^2(\sigma-\theta)}\Big]$$

$$Q_2=\frac{M\pi(M)\,\alpha(M_S)\,\phi_S^{1/(\gamma-1)}}{\alpha(M)\,\pi(M_S)\,\tau(M_S)}\Big[\cos(2\sigma-\theta)\left(\frac{1}{f}-1\right)$$

$$+\frac{4\sin\sigma\cos^2(\sigma-\theta)}{f^2(\gamma+1)M^2\sin^3(\sigma-\theta)}\Big]$$

$$Q_3=\frac{M\pi(M)\,\alpha(M_S)\,\phi_S^{1/(\gamma-1)}}{\alpha(M)\,\pi(M_S)\,\tau(M_S)}\Big[\frac{\sin\sigma\sin(\sigma-\theta)}{f}+\frac{\cos(\sigma-\theta)\cos\sigma}{1}$$

$$-\frac{4\sin\sigma\cos^2(\sigma-\theta)}{f^2(\gamma+1)M^2\sin^3(\sigma-\theta)}\Big]$$

$$Q_4=\left(\frac{1}{r}-\frac{\cot\sigma}{\varepsilon}\right)\frac{M_S}{\tau(M_S)}+\frac{M\pi(M)\,\alpha(M_S)\,\phi_S^{1/(\gamma-1)}}{\alpha(M)\,\pi(M_S)\,\tau(M_S)}$$

$$\times\Big[\left(\frac{1}{r}+\frac{\cot\sigma}{\varepsilon}\right)\Big\{\frac{\cos(\sigma-\theta)\sin\sigma}{f}-\frac{\sin(\sigma-\theta)\cos\sigma}{1}\Big\}$$

$$-\frac{2}{\varepsilon}\Big\{\frac{\cos(\sigma-\theta)\cos\sigma}{f}+\frac{\sin(\sigma-\theta)\sin\sigma}{1}\Big\}\Big]$$

$$\tau(M)=\left(1+\frac{\gamma-1}{2}M^2\right)^{-1},\quad \pi(M)=\left(1+\frac{\gamma-1}{2}M^2\right)^{-\gamma/(\gamma-1)}$$

The derivatives $\frac{dM}{dr}$ and $\frac{d\theta}{dr}$ are obtained from (6). Thus, the final system of equations to be solved are as follows :

$$\frac{d\theta}{dr}=\frac{L-\varepsilon-r\cot\sigma}{(L-\varepsilon)^2+r^2} \qquad \dots \quad (13)$$

$$\frac{dM}{dr}=-\frac{2M\,[r+(C-\varepsilon)\cot\sigma]}{\tau(M)\,(1-M^2)\,(\eta+\sqrt{Q})\,\eta} \qquad \dots \quad (14)$$

$$\frac{d\sigma}{dr}=-\left(P_1\frac{dM}{dr}+P_3\frac{d\theta}{dr}+P_4\right)\Big/P_2 \qquad \dots \quad (15)$$

$$\frac{dM_s}{dr}=-\left(Q_1\frac{dM}{dr}+Q_2\frac{d\sigma}{dr}+Q_3\frac{d\theta}{dr}+Q_4\right)\Big/(1-M_S^2) \qquad \dots \quad (16)$$

$$\frac{d\varepsilon}{dr}=-\cot\sigma \qquad \dots \quad (17)$$

Equation (12) remains same for both parabolic and linear laws, while equation (11) differs only with respect to the coefficient P_4 which is given for linear law as

$$P_4 = \frac{1}{2}\left(\frac{1}{r} + \frac{\cot\sigma}{\varepsilon}\right)\left[\frac{\cos^2(\sigma-\theta)\sin 2\sigma}{f} - \frac{\cos 2\sigma \sin 2(\sigma-\theta)}{1}\right.$$

$$\left. - \frac{f\sin^2(\sigma-\theta)\sin 2\sigma}{1}\right] - \frac{2}{\varepsilon}\left[\frac{\cos^2(\sigma-\theta)\cos^2\sigma}{f}\right.$$

$$\left. + f\frac{\sin^2(\sigma-\theta)\sin^2\sigma}{1} + \frac{\sin 2(\sigma-\theta)\sin 2\sigma}{2}\right]$$

$$- \frac{2}{\varepsilon M^2\gamma}\left[\frac{2\gamma}{\gamma+1}M^2\sin^2(\sigma-\theta) - \frac{\gamma-1}{\gamma+1} - \frac{\pi(M_S)\,\phi_S^{-1/(\gamma-1)}}{\pi(M)}\right].$$

On the line of symmetry $r=0$, the coefficients P_4 and Q_4 becomes indeterminate and hence are defined using L' Hospital's rule. Therefore at $r=0$ for parabolic law,

$$P_4 = -\frac{12}{5\varepsilon}\left[\frac{2+(\gamma-1)M^2}{(\gamma+1)M^2}\right] - \frac{12}{5\varepsilon\gamma M^2}\left[\frac{2\gamma M^2}{\gamma+1} - \frac{\gamma-1}{\gamma+1}\right.$$

$$\left. - \frac{\phi_S^{-1/(\gamma-1)}\,\pi(M_S)}{\pi(M)}\right]$$

and

$$Q_4 = -\frac{2}{\varepsilon}\,\frac{M\pi(M)\,\alpha(M_S)\,\phi_S^{-1/(\gamma-1)}}{\alpha(M)\,\pi(M_S)\,\tau(M_S)}$$

where θ the slope angle of the velocity vector to the Z-axis, σ-shock angle, ε- the shock stand off distance, M-Mach number are the dependent variables. In deriving the set of equations we have followed closely the analysis developed by Belov *et al* (1973).

Boundary conditions : Now appropriate boundary conditions are to be specified for ϕ, θ, σ, ε and M_S to slove the first order coupled ordinary differential equations (13) to (17). The boundary conditions imposed on the booster surface and the axis of symmetry are

$$\text{at } Z=0, \quad \phi=\phi_0, \quad v=0$$

$$\text{at } r=0 \quad \phi=\phi_0 \quad \sigma=\pi/2$$

$$\varepsilon=\varepsilon_0 \quad u=0$$

In order to determine $M_s(r)$, $\sigma(r)$, $\varepsilon(r)$ upto the sonic corner, the initial conditions for the equations (13) to (17) are at $r=0$, $M(o)=0$, $\sigma(o)=\pi/2$, $\varepsilon(o)=\varepsilon_0$, $\theta=0$. Since the bow shock is a detached one, the shock stand off distance ε_0 is a free parameter. The integration of equations (13) to (17) starts at $r=0$ and terminates at $r=R$ (the edge of the booster).

Determination of ε_0 : The initial value of the shock stand off distance ε_0 is unknown and must be chosen so that the singular sonic point occurs at the corner of the booster.

A halving mode iteration scheme between successively improved upper and lower bounds of ε_0 have been incorporated into our Computer programme for computing an acceptable value of ε_0 (accurate to eight decimal places). The appropriate bounds for ε_0 are taken from Belotserkovskii [1958]. This is a very sensitive parameter for, if ε_0 is slightly larger or slightly small, the sonic singularity occurs down-stream or upstream from the corner. Values of ε_0 different from exact one may yield solutions which have a behaviour totally different from the true solution. Similar iteration scheme has been used by Xarikos and Anderson [1962] in their investigation of uniform flow past blunt bodies and also by Gummer and Hunt [1974].

3. **Integration procedure :** The numerical integration of the system of ordinary differential equations (13) to (17) has been initiated with the following initial values of the dependent variables.

$M_s=0$, $\sigma=\pi/2$, $\theta=0$, $\varepsilon=\varepsilon_0$ (some approximated known value) at the axis $r=0$. The initial value of $\frac{d\sigma}{dr}$ and $\frac{dM}{dr}$ are indeterminate at the axis and are obtained by L' Hospital's rule. However, the shock stand off distance ε_0 is unknown and must be determined as a part of the solution. The problem then reduces to a two point boundary value problem which is to be solved iteratively to get physically meaningful solution at the corner of the booster sonic point. Basically the iterative procedure involves successive refinement of ε_0 such that the denominator of equation (16) vanishes identically at the sonic corner. A discontinuity in the velocity gradient at this point is consistent with the requirement of the inviscid theory and therefore ε_0 is varied until $M_s=1$ at the corner of the booster. The actual integration has been carried out using fourth order Runge-Kutta method. Flow field computations have been carried out with the known nozzle exit conditions and for various nozzle to booster surface distances assuming that booster is normal to and within the first cell of the jet plume.

Pressure and density distributions on the surface of the booster have been computed from the following expressions :

$$P_s=[\phi_s]^{-\frac{1}{(\gamma-1)}} \cdot \tau[M_s(\gamma)]^{\frac{\gamma}{(\gamma-1)}}$$

$$\rho_s=[\phi_s]^{-\frac{1}{(\gamma-1)}} \cdot \tau[M_s(\gamma)]^{\frac{1}{(\gamma-1)}}$$

and the results obtained are presented in the next section.

By using source flow approximation, fairly acceptable solutions are obtained in the far field. It may be noted that when the booster is very close to the sustainer, the source flow approximation will not yield any meaningful results. Besides no experimental facility is available to the authors for comparison.

4. Results and Discussion : In our present investigation we have considered the following parameters :

(i) $\gamma \quad = \quad 1.25$

(ii) $M_B \quad = \quad 3.9$

(iii) $\theta_N \quad = \quad 15°$

(iv) $R_B \quad = \quad 0.445$ m

(v) $R \quad = \quad 0.5000$ m

(vi) $n \quad = \quad 5$

The separation distance considered are 12 and 20. Our Computer program has been checked for consistency of the results by varying the mesh size and after few preliminary experiments we fixed the value of mesh ratio as $h=\bar{R}/200=(R/Re)/200$. For a given separation distance, we have computed pressure, Mach number and shock stand off distance. Their distributions on the surface of the booster have been shown graphically. Figures 2 and 3 depict the distribution of pressure on the surface of the booster for separation distances $L=12$ and $L=20$ respectively. In figure 2, we have considered that normal component of the velocity to vary both by parabolic and linear law while the radial velocity varies only linearly. It is found that there is a change in the peak pressure by 15% while the pressure distribution has the same qualitative behaviour along the surface of the booster.

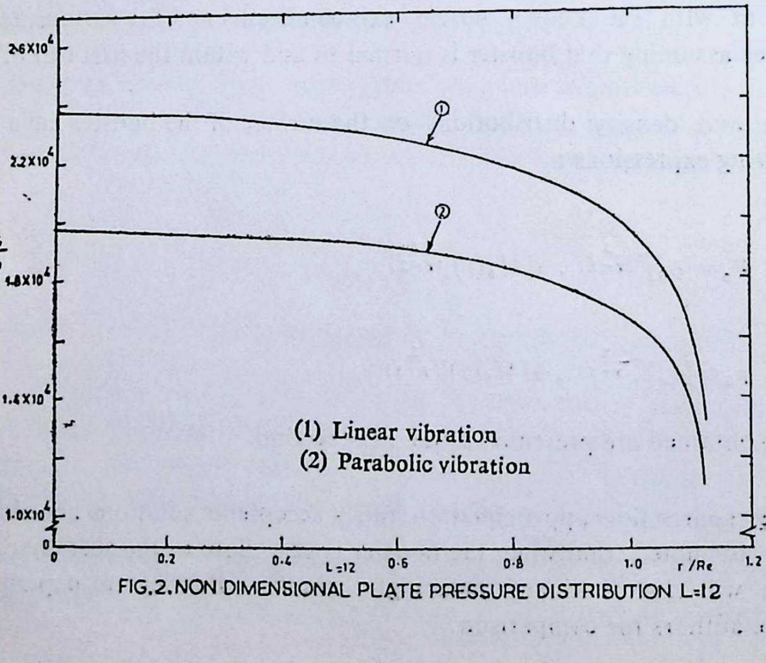


FIG. 2. NON DIMENSIONAL PLATE PRESSURE DISTRIBUTION L=12

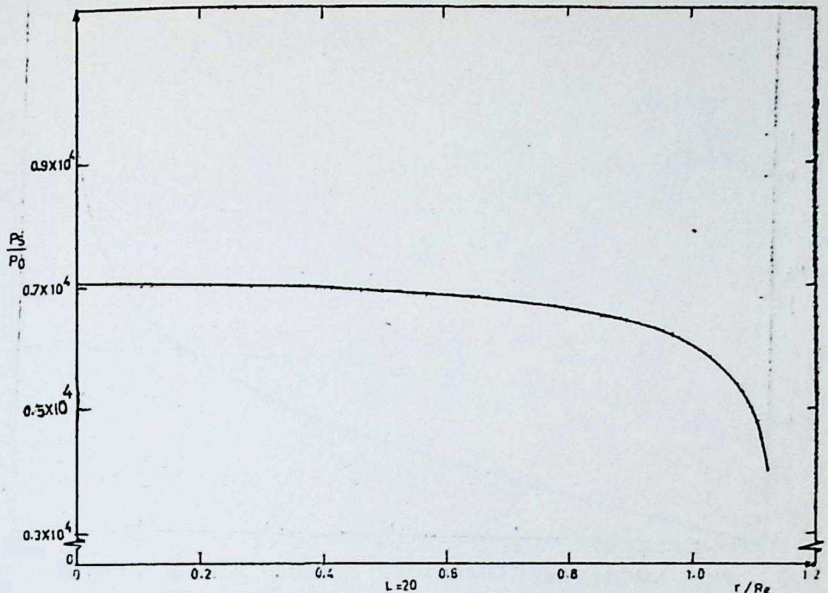


FIG.3. NON DIMENSIONAL PLATE PRESSURE DISTRIBUTION L=20

Mach number distribution on the surface of the booster for separation distance $L=12$ is shown in Figure 4 with the variation of normal component of the velocity both by parabolic and linear laws.

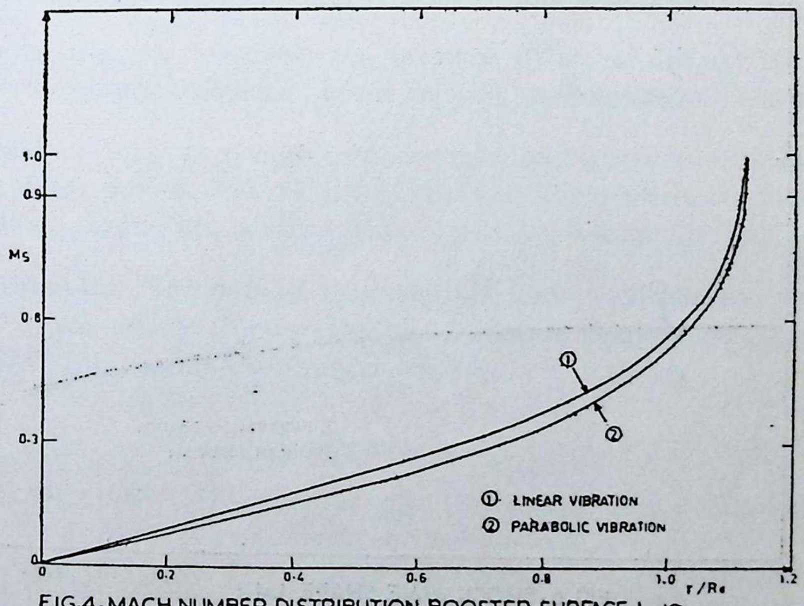


FIG.4. MACH NUMBER DISTRIBUTION BOOSTER SURFACE L=12

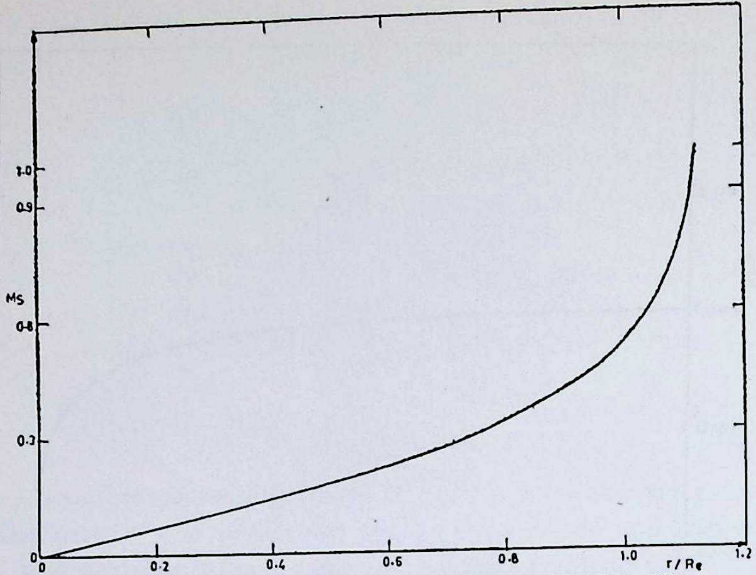


FIG. 5. MACH NUMBER DISTRIBUTION BOOSTER SURFACE L = 20

The shape of the bow shock is shown in Figures 6 and 7 which is concave towards the booster but is very nearly flat. This behaviour can be attributed to the non uniformities of the flow. It is also observed that the shock stand off distance decreases with the decrease in separation distance, while the value of ε_0 is differing only in the third decimal place where the variation in the normal component of this velocity is assumed to follow both parabolic and linear law.

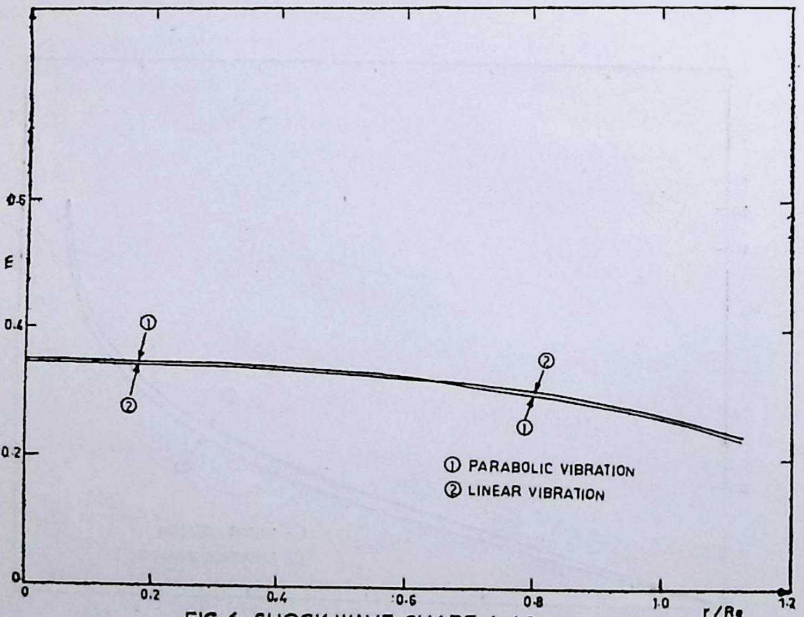


FIG. 6. SHOCK WAVE SHAPE L = 1.2

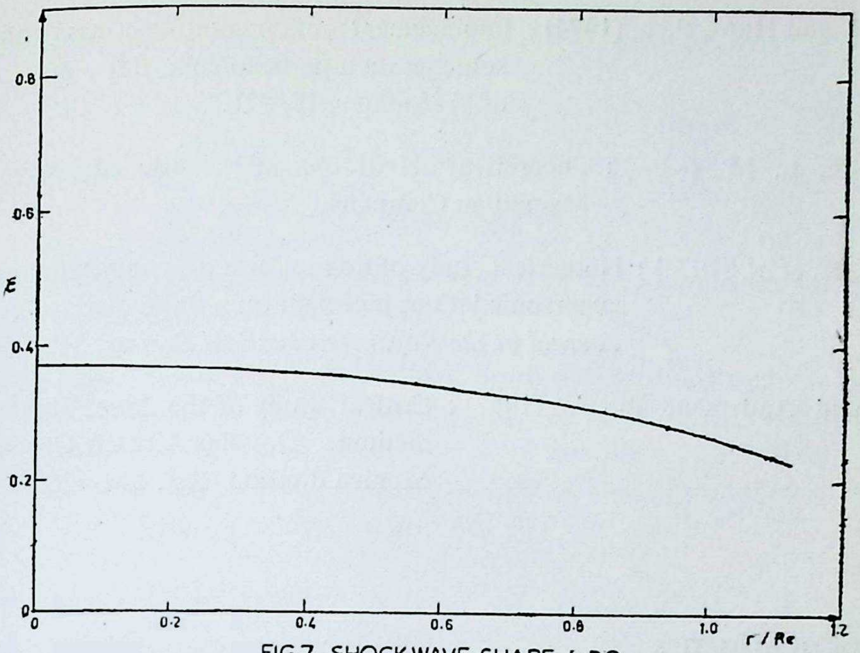


FIG 7 SHOCK WAVE. SHAPE L=20

Conclusions : Some of the observations are :

(i) the peak pressure occurs on the axis of symmetry, and decreases in a smooth continuous manner.

(ii) the peak pressure decreases with the increase of separation distance.

(iii) As the booster edge is approached, there is a steep fall in pressure.

(iv) As the radial distance increases, the pressure difference due to parabolic and linear variation in the normal component of the velocity has decreased.

The method of integral relations with a single strip approximation and also with the assumption that both normal and radial velocities vary only linearly can predict reasonably accurate flow field on the booster surface when a supersonic sustainer jet impinges on it.

Acknowledgement : The authors are grateful to Prof. P. Sivalingam, Vice-Chancellor, Anna University, Prof. W. P. Vijayaraghavan, Principal, College of Engineering and Prof. S. Vaidynathan for their kind encouragement.

References

Belotserkovskii, O. M. (1958) : Flow past a circular cylinder with a detached shock wave. *Vychislitelnaia mathematika*, 3, 149.

Belov, I. A., *et al.* (1973) : Supersonic Underexpanded jet impingement upon flat plate. *Int. J. Heat and mass transfer*, **16**, 2067.

Gummer, J. H. and Hunt, B. L. (1974) : Impingement of nonuniform axisymmetric supersonic jet on a perpendicular flat plate. *Israil Journal of Technology*, **12**, 221.

Milne-Thomson, L. M. (——) : Theoretical Hydrodynamics, 4th ed. New York. The Macmillan Company.

Sankara Rao, K. *et al.* (1976) : Numerical study of nonuniform axisymmetric under expanded supersonic jet impingement on a finite disc. *The Aeronautical Journal of the Royal Aeronautical Society*, 313.

Xerikos, J. and Anderson, W. A. (1962) : Critical study of the direct blunt body integral method. Douglas Aircraft Corporation, Santa Monica division, Ref. SM-42603.

DEPARTMENT OF MATHEMATICS
COLLEGE OF ENGINEERING
ANNA UNIVERSITY,
MADRAS 600 025, INDIA

Bull. Cal. Math. Soc., 75, 317-319 (1983)

CURVATURE AND COVERING THEOREMS FOR UNIVALENT FUNCTIONS

V. KARUNAKARAN*

(*Received 24 September 1979 and Revision Received 27 November 1979*)

Let f be regular and univalent in the unit disc E of the complex plane. Suppose $c \varepsilon E$ and there is a smooth curve passing through $f(c)$ and completely contained in the image domain. We propose to obtain a lower bound for the radius of curvature of its preimage at c. We deduce a number of simple consequences one of which generalises a beautiful result due to Caratheodary [Ahlfors 1973, Theorem 1-4, p. 7].

We need the following lemma.

LEMMA : *Let f be regular and univalent in E and $c \varepsilon E$. Let a smooth curve Γ through $f(c)$ lying entirely in the image domain have a curvature k_1 at $f(c)$. Then the radius of curvature $1/|k|$ of the preimage γ of Γ at $z=c$ has β as a lower bound where*

$$\beta^{-1}=\{ |k_1| |f'(c)| + |f''(c)| / |f'(c)| \}$$

Proof: If γ is represented by $z=z(t)$ and Γ by $w=w(t)=f(z(t))$ then by the well-known formula for the curvature we have

$$k_1 = d \arg df(z(t)) / |dw|$$

and

$$k = d \arg d\, z(t) / |dz|$$

at the parametric point t.

Thus at any parametric point t we have

$$k_1 = d \arg f'(z(t) / |dw| + d \arg z'(t) \,|\, |dw|$$

$$= d\, \mathrm{Im} \log f'(z(t) / |dw| + k\, |dz / dw|$$

Hence if we denote by k_1 and k the curvatures of Γ and γ at the relevant parametric point t for which $c=z(t)$ and $f(c)=w(t)$ we have

---

* Research supported by the UGC Grant F. 25-3(8772)77 (SRI)

$$|k| \leqslant |k_1|\,|dw/dz|_{z=o} + |d\,\mathrm{Im}\log f'(z)/dz|_{z=c}$$

$$\leqslant |k_1|\,|f'(c)| + |f''(c)\,/\,f'(c)| \qquad \cdots \quad (1)$$

The lemma is an immediate consequence of the above inequality.

THEOREM. *Let* $f(z)=z+a_2z^2+\ldots$, *be univalent and regular in* E *with* $|a_2| \leqslant 1$. *If there exists a smooth curve* γ *through the origin which is mapped onto a curve* Γ *whose curvature at the origin is bounded above in absolute value by* $2(1-|a_2|)$ *then the circle of curvature at any point of* γ *meets* $|z|=1$.

Proof: Using inequality (1) of the lemma with $c=o$ we have

$$|k| \leqslant |k_1| + 2\,|a_2| \leqslant 2$$

Thus the circle of curvature at the origin for γ meets $|z|=1$. This result has an invariant formulation. In the words of Ahlfors we have "The circle of curvature is the circle of highest contact. A conformal self mapping of the unit disc preserves circles, and order of contact. Circles of curvature are mapped on circles of curvature and our result not only holds at origin but at any point of γ as well". The proof of the theorem is now complete.

Remarks 1. In case f is univalent with $|a_2| \leqslant 1$ and maps a smooth arc γ through the origin onto a straight line then $k_1=0$ and the theorem applies to give that the circles of curvature of γ meet $|z|=1$. This is a slight generalisation of the result of Caratheodary referred to above.

2. The conclusion of the theorem remains valid if f is convex in the unit disc and the image domain containing a circle Γ of radius at least $2^{-1}(1-|a_2|)^{-1}$ through the origin and γ is the preimage of Γ. In this case the image domain covers the disc whose boundary is Γ since f (being univalent and regular is consequently a homeomorphism) maps the simply connected domain E onto a simply connected domain.

3. If the image domain covers a disc of radius x through $f(c)$ where f is a regular univalent function in E and $c \in E$ then we can find a lower bound for the radius of curvature at c of the preimage of the boundary of this disc purely in terms of $|c|$. For this we need only observe that

$$|f'(c)| \leqslant (1+|c|)\,/\,(1-|c|)^3$$

and

$$|f''(c)\,/\,f'(c)| \leqslant (4+2|c|)\,/\,(1-|c|^2).$$

The lower bound needed will then be

$$x(1-|c|^3\,(1+|c|)\,/\,\{\,(1+|c|)^2+x(4+2|c|)\,(1-|c|)^2\,\}.$$

Reference

Ahlfors, L. V., (1973) : Conformal Invariants, Topics in Geometric function theory, Mcgraw-Hill.

DEPARTMENT OF MATHEMATICS
NATIONAL COLLEGE
TIRUCHIRAPALLI 620 001, INDIA.
Current address of the author :
SCHOOL OF MATHEMATICS,
MADURAI UNIVERSITY,
MADURAI-625 021, INDIA.

References

Abbot, L. E. (19[illegible]): Contextual Invariants, [illegible]

[illegible] MATHEMATICS,
[illegible]
[illegible] BANGALORE 560 001, INDIA

Current address of the author:

SCHOOL OF MATHEMATICS,
MADURAI UNIVERSITY,
MADURAI-625 021, INDIA.

Bull. Cal. Math. Soc., 75, 321-329 (1983)

MAGNETOHYDRODYNAMIC BOUNDARY LAYER WITH SUCTION ALONG A POROUS WALL IN HOWARTH'S FLOW

SUTAPA MUKHERJEE and R. C. CHOUDHARY

(*Received December 5, 1978*)

Abstract : A solution has been obtained for the laminar magnetohydrodynamic boundary layer of an incompressible fluid of small electrical conductivity along a porous wall in Howarth's flow with uniform suction and uniform magnetic field perpendicular to the wall. The momentum and the kinetic energy integral equations and the wall compatibility condition have been used to find an approximate solution with the help of Pohlhausen's fourth degree velocity profiles.

1. Introduction : Howarth (1938) ttudied the boundary layer of an incompressible and non-conducting fluid for the potential flow given by $U(x) = U_0\left(1-\frac{x}{L}\right)$ and found that separation occurred at $\frac{x}{L} = 0.120$. Leigh (1955) investigated the same problem by using an automatic computer and found the point of separation at $\frac{x}{L}=0.1198$. Curle (1960) has suggested that for Howarth's flow along a porous wall separation can be completely avoided if the suction velocity v_s is given by $\frac{v_s}{\sqrt{-\nu U'}} > 1.55$.

In this paper an investigation has been made into the laminar incompressible boundary layer of a fluid of small electrical conductivity with uniform suction in the presence of uniform transverse magnetic field along a porous wall of Howarth's flow. The approximate method of solution given by Pohlhausen (1921) for non conducting fluids and improved by Head (1957) to suit to the problems with suction has been extended here to the problems of MHD boundary layers with suction in the presence of transverse magnetic field. The momentum and the kinetic energy integral equations deduced for the problems of MHD boundary layers with suction have been numerically solved for different values of suction and magnetic field strength parameters to illustrate the combined effect of suction and transverse magnetic field on the MHD boundary layer.

2. Equations : Following Head (1957) and using the MHD boundary layer equation the momentum and the kinetic energy integral equations for MHD boundary layers along porous walls can be rederived.

(a) **Momentum Integral Equation :** With x as the coordinate along the wall and y the coordinator perpendicular to it, the two dimensional boundary layer equation for an incompressible fluid of small electrical conductivity in the presence of an externally applied transverse magnetic field of uniform strength H_0 is (Pai 1962),

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma}{\rho} \mu_e^2 H_0^2 u \qquad \dots \quad (1)$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \dots \quad (2)$$

and for uniformly porous walls the boundary conditions are

$$\begin{aligned} y = 0 \ &: u = 0,\ v = v_s\ ; \\ y = \infty\ &: u = U(x) \end{aligned} \qquad \dots \quad (3)$$

The density ρ, the pressure p, the viscosity μ, the magnetic permeability μ_e and the electrical conductivity σ are assumed constants throughout the flow field. The induced magnetic field has been omitted in comparison with the applied magnetic field.

Because of the condition at $y = \infty$, we have from equation (1),

$$- \frac{1}{\rho} \frac{\partial p}{\partial x} = U \frac{dU}{dx} + \frac{\sigma}{\rho} \mu_e^2 H_0^2 U$$

Hence eqn. (1) becomes,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} + \frac{\sigma}{\rho} \mu_e^2 H_0^2 (U - u) \qquad \dots \quad (4)$$

Integrating equation (4) with respect to y from $y = 0$ to $y = \infty$ under the boundary conditions (3) we get

$$\frac{d\theta}{dx} = \frac{\tau_0}{\rho U^2} - \frac{\theta}{U} \frac{dU}{dx} \left(2 + \frac{\delta^*}{\theta} \right) + \frac{v_s}{U} - \frac{m_1 \delta^*}{U} \qquad \dots \quad (5)$$

where $\theta = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U} \right) dy,$

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U} \right) dy,$$

and $$m_1 = \frac{\sigma}{\rho}\, \mu_e^2\, H_0^2$$

Hence, the momentum integral equation in dimensionless form is

$$\frac{dt^*}{d\bar{x}} = \frac{2}{\bar{U}}\, [\, l-(2+H)\,\Delta - Ht^*\, R_m^2+\lambda] \qquad \dots \quad (6)$$

where $$\bar{x} = \frac{x}{L}\, ,\; \bar{U} = \frac{U}{U_0}\, ,$$

$$t^* = \left(\frac{\theta}{L}\right)^2 \frac{U_0 L}{\nu}\, ,$$

$$l = \frac{\theta}{U}\left(\frac{\partial u}{\partial y}\right) y = 0,$$

$$H = \frac{\delta^*}{\theta}\, ,$$

$$\Delta = \frac{\theta^2}{\nu}\, \frac{dU}{dx}\, ,$$

$$R_m^2 = \frac{\sigma\, \mu_e^2\, H_0^2\, L}{\rho U_0}\, ,$$

$$\lambda = \frac{v_s \theta}{\nu}\, ;\; \lambda > 0 : \text{injection}$$

$$\lambda < 0 : \text{suction.}$$

(b) Kinetic energy integral equation : Adding $\frac{u}{2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$ to the left hand side of equation (4), multiplying through by u and integrating with respect to y from $y = 0$ to $y = \propto$ under the boundary conditions (3) we have

$$\frac{d\epsilon}{dx} = \frac{2\nu}{\theta U}\, D - \frac{2m_1\theta}{U} - \frac{3\epsilon}{U}\frac{dU}{dx}+\frac{v_s}{U} \qquad \dots \quad (7)$$

where $$\epsilon = \int_0^\infty \frac{u}{U}\left[\, 1 \left(\frac{u}{\,}\right)^2\right] dy,$$

and $$D = \int_0^\infty \left(\frac{\theta}{U}\right)^2 \left(\frac{\partial u}{\partial y}\right)^2 \frac{dv}{\sigma}$$

$$i.\,e.\quad \frac{d}{dx}\left(\frac{\epsilon^2}{\nu}=\frac{2H\epsilon}{U}\right)[2D-3H_\epsilon\Delta-2t^*_{*}\,R_m^2+\lambda] \qquad \cdots\ (8)$$

where $$H_\epsilon=\frac{\epsilon}{\theta}\,.$$

The direct variation of H_ϵ with respect to x is given by

$$\frac{dH_\epsilon}{dx}=\frac{1}{\theta}\frac{d\epsilon}{dx}-\frac{\epsilon}{\theta^2}\frac{d\theta}{dx} \qquad \cdots\ (9)$$

Substituting for $\frac{d\epsilon}{dx}$ and $\frac{d\theta}{dx}$ from equation (7) and (5) respectively into equation (9) we have the kinetic energy integral equation in dimensionless form,

$$\frac{dH_\epsilon}{d\bar{x}}=\frac{1}{\bar{U}t*}\,[2D-H_\epsilon\{l-(H-1)\,\Lambda-Ht*\,R_m^2+\lambda\}-2t*\,R_m^2+\lambda] \qquad \cdots\ (10)$$

(c) **Wall compatibility condition :** At the wall,

$$y=0:u=0,\ v=v_s\ ;$$

Hence equation (4) gives,

$$\nu\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}=-U\frac{dU}{dx}-\frac{\sigma}{\rho}\mu_e^2H_0^2\,U+v_s\left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$i.\,e.\quad m=-\Lambda-R_m^2\,t*+\lambda\,l \qquad \cdots\ (11)$$

where $$m=\frac{\theta^2}{U}\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}.$$

The two integral equations (6) and (10) and the wall compatibility condition (11) may be used with the help of a presupposed family of velocity profiles for the approximate calculation of the hydromagnetic boundary layers along porous walls in the presence of transverse magnetic field.

3. **Velocity profiles :** With the modification that the local boundary layer thickness would now depend on the rate of suction and the strength of magnetic field Pohlhausen's fourth degree polynomial velocity profiles may be used to investigate the MHD boundary layers with suction.

The profiles are given by

$$\frac{u}{U}=F(\eta)+K\,G(\eta),\ \eta=\frac{y}{\delta(x,\,\bar{v}_s,\,R_m^2)} \qquad \cdots\ (12)$$

where
$$F(\eta) = 2\eta+2\eta^3+\eta^4$$

$$G(\eta) = \frac{1}{6}\left[\eta - 3\eta^2+3\eta^3 - \eta^4\right]$$

and K is a free parameter which is used to satisfy the additional condition at the wall. For this system of velocity profiles the wall compatibility conditions (11) becomes

$$K\left(\frac{\theta^2}{\delta^2}\right) = -\lambda l+\Lambda+R_m^2\, t_* \qquad \dots \quad (13)$$

The compatibility condition (11) will be satisfied in course of solution with the help of the profile parameter. The profile parameter K is now related with the pressure gradient parameter Λ, the suction parameter λ and the magnetic field strength R_m^2 as stated by the wall compatiblity condition (13). For this family of velocity profiles the variations of the boundary layer parameters H, H_ϵ, l and D against the profile parameter K are shown in table 1.

4. **Howarth's flow :** It is proposed to investigate the boundary layer with suction in the presence of transverse magnetic field along a porous wall in Howarth's flow for which the potential flow velocity is given by $U(x) = U_0\left(1 - \frac{x}{L}\right)$. The MHD boundary layer is investigated along one wall from the entrance section $(x = 0)$ of the diverging channel where the velocity is U_0. The wall is supposed to be porous and the magnetic field is perpendicular to it. For this type of flow

$$\frac{U(x)}{U_0} = 1 - \frac{x}{L}$$

i. e.
$$\bar{U} = 1 - \bar{x}$$

$$\Lambda = t_* \ \frac{d\bar{U}}{dx} = - t_*$$

and
$$\lambda = \frac{v_s\theta}{\nu} = t_*^{1/2}\ \bar{v}_s;$$

where
$$\bar{v}_s = \frac{v_s}{U_0}\sqrt{\frac{U_0 L}{\nu}}$$

Then equations (6), (10) and (11) reduce to

$$\frac{dt_*}{d\bar{x}} = f(x,\, t_*,\, H_\epsilon)\,,$$

where
$$f(\bar{x},\, t_*,\, H_\epsilon) = \frac{2}{1-\bar{x}}\left[l+(2+H)\, t_* - H t_*\, R_m^2+\bar{v}_s t_*^{1/2}\right] \qquad \dots \quad (14)$$

$$\frac{dH_e}{d\bar{x}} = g(\bar{x}, t_*, H_e),$$

where
$$g(\bar{x}, t_*, H_e) = \frac{1}{(1-\bar{x})t_*}[2D - H_e\{l+(H-1)\, t_* - Ht_*\, R_m^2+\bar{v}_s t_*^{1/2}\} - 2t_*\, R_m^2+\bar{v}_s t_*^{1/2}] \quad \ldots \quad (15)$$

and
$$m = (1 - R_m^2)\, t_*+\bar{v}_s\, t_*^{1/2} \quad \ldots \quad (16)$$

TABLE—1 BOUNDARY LAYER CHARACTERISTICS FOR POHLHAUSEN'S VELOCITY PROFILES AGAINST THE PROFILE PARAMETER K (REF. 8)

K	M	H	H_e		D
7.5	−0.0800	2.300	1.605	0.336	0.1962
7.052	−0.0770	2.308	1.603	0.332	0.1950
7	−0.0767	2.309	1.603	0.331	0.1948
6	−0.0689	2.333	1.599	0.321	0.1922
5	−0.0599	2.361	1.595	0.310	0.1894
4	−0.0497	2.392	1.591	0.297	0.1864
3	−0.0385	2.427	1.586	0.283	0.1834
2	−0.0264	2.466	1.581	0.268	0.1805
1	−0.0135	2.508	1.576	0.252	0.1775
0	0.0000	2.554	1.571	0.235	0.1745
− 1	0.0140	2.604	1.566	0.217	0.1718
− 2	0.0284	2.647	1.561	0.199	0.1690
− 3	0.0429	2.716	1.556	0.179	0.1667
− 5	0.0720	2.847	1.547	0.140	0.1627
− 7	0.0999	2.999	1.540	0.100	0.1596
− 9	0.1254	3.176	1.534	0.059	0.1576
−11	0.1474	3.383	1.529	0.019	0.1570
−12 (Separation)	0.1567	3.500	1.528	0.000	0.1568

5. **Solution :** The boundary layer begins to develop from the leading edge of the wall ($\bar{x} = 0$) which is taken as the starting point for the approximate step by step calculation.

At the starting point $t_* = 0$ and from equation (16) we have for all values R_m^2 and $\bar{v}_s$,

$$m = 0$$

Hence from table 1,

$$H = 2.554$$

$$H_e = 1.571$$

$$l = 0.235$$

$$D = 0.1745$$

The kinetic energy integral equation (15) exhibits singularity at the leading edge ($\bar{x} = 0$). Therefore, for a short distance from the edge the momentum integral equation (14) and the wall compatibility condition (16) have been solved by Runge-Kutta method with the help of table 1 [Ref. 8]. After a few steps the momentum and the kinetic energy integral equations have been solved simultaneously by Runge-Kutta method. As Runge-Kutta method is too laborious the momentum and the kinetic energy integral equations have been integrated further downstrem by Adam's method using a quadrature formula with repeated ascending differences.

6. **Results :** Calculations have been made for different constant values of the suction parameter $\bar{v}_s$ and the magnetic field strength parameter R_m^2. The results of calculation have been represented by curves in figures 1 and 2. It is observed that the application of suction as well as the transverse magnetic field reduces the boundary layer thickness and shits the point of separation downstream. The effect of suction on the MHD boundary layer in the presence of transverse magnetic field has been shown in table 2.

TABLE—2

Magnetic field strength $R_m^2 = \frac{\sigma}{PU_0} \mu_e^2 H_0^2 L$	Suction $\bar{v}_s = \frac{v_s}{U_0}\sqrt{\frac{U_0 L}{\nu}}$	Point of separation $\bar{x}_s$
0.0	0.0	0.1248
0.1	—0.1	0.1433
0.0	—0.3	0.1494
0.3	0.0	0.1583
0.3	—0.3	0.1920

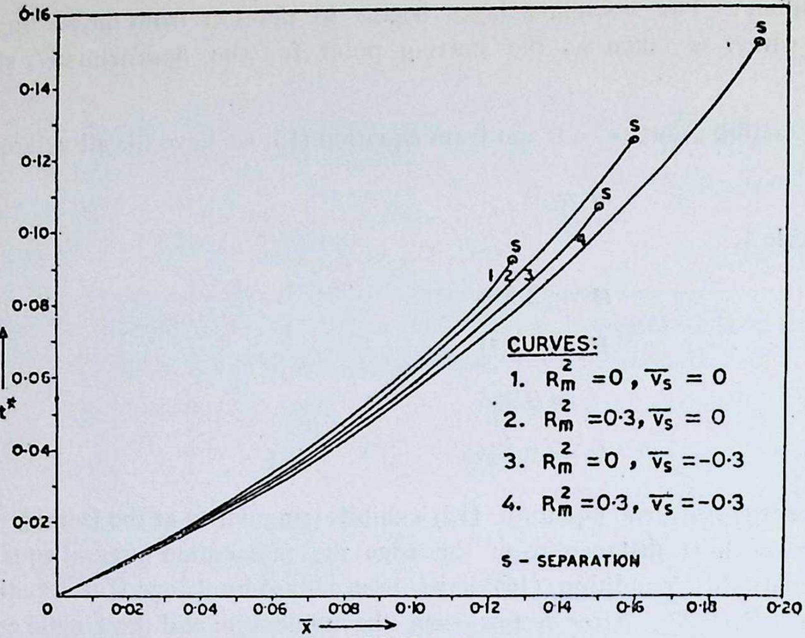
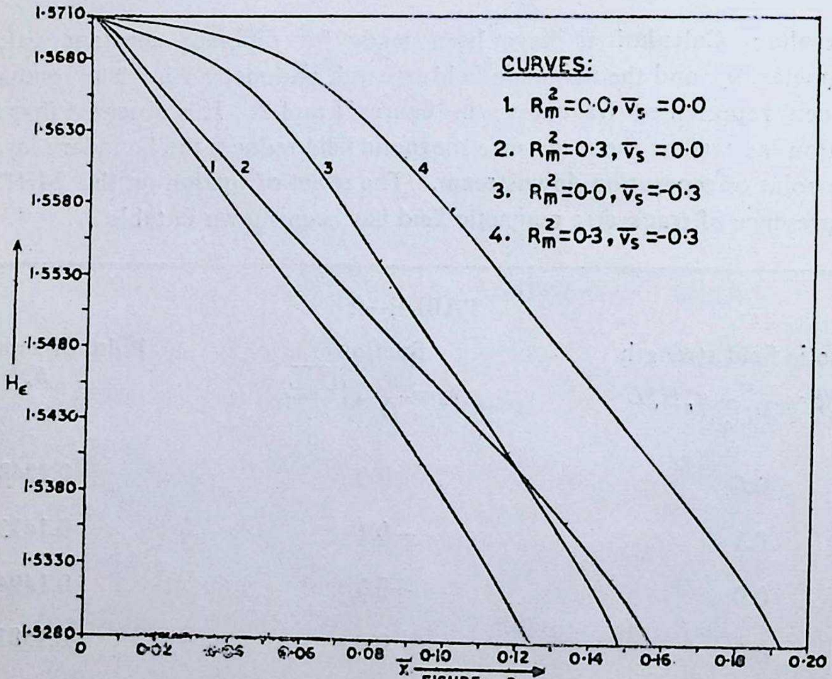


FIGURE – 1

MHD BOUNDARY LAYER WITH SUCTION ALONG A POROUS WALL IN HOWARTH'S FLOW. VARIATION OF t^* AGAINST $\overline{x}$ FOR DIFFERENT VALUES OF R_m^2 AND $\overline{v_s}$.



FIGURE– 2

MHD BOUNDARY LAYER WITH SUCTION ALONG A POROUS WALL IN HOWARTH'S FLOW. VARIATION OF H_ϵ AGAINST $\overline{x}$ FOR DIFFERENT VALUES OF R_m^2 AND $\overline{v_s}$.

For $R_m^2 = 0$, $\bar{v}_s = 0$ the equations reduce to the equations for the boundary layer of a non-conducting fluid along a solid wall and the point of separation obtained by the present method is $\bar{x}_s = 0.1248$ which is very close to the point of separation $\bar{x}_s = 0.120$ obtained by Howarth (1938) by an exact method of solution. It is expected that the results obtained by the present approximate method are sufficiently accurate to describe the effect of suction on the MHD boundary layer along a porous wall in Howarth's flow in the presence of transverse magnetic field.

References

Bennet, A., Milne, W. E. and Batman, H. : Numerical integration of differential equations. Dover Publications, New Work, p. 73.

Curle, N. (1960) : The estimation of laminar skin friction including the effect of distributed suction ANO. **11**.

Howarth, L. (1938) : On the calculation of laminar boundary layer equation. *Proc. Roy. Soc. London* **1**. 164.

Head, M. R. (1957) : An approximate method of calculating the boundary layer in two dimensional incompressible flow. *ARC.*, MR. 3123.

Leigh, D. C. F. (1955) : Laminar boundary layer equation. *Proc. Camb. Phil. Soc.* **51**.

Pohlhausen, K. (1921) : Zur naherungsweisen integration der differentialgleichung der laminaren Reibungsschieht. *ZAMM* **1**, 252.

Pai, S. I. (1962) : Magnetogasdynamics and Plasma dynamics. *Wien Springer-Verlag*. Austria 66.

Scarborough, J. B. (1966) : Numerical Mathematical Analysis. Johns Hopkins Press, Baltimore.

Schuh, H. (1953) : Calculation of unsteady boundary layers in two dimensional laminar flow. *ZFW*. **1**, 122.

C. S. I. R. FELLOW,
DEPARTMENT OF MATHEMATICS
RANCHI UNIVERSITY,
RANCHI-834 008
and
PROFESSOR OF MATHEMATICS,
DEPARTMENT OF MATHEMATICS,
RANCHI UNIVERSITY,
RANCHI-834 008.

Bull. Cal. Math. **75**, 331-338 (1983)

PSEUDO-IDEALS IN SOME FINITE GROUPS AND IN FINITELY GENERATED ABELIAN GROUPS

T. K. DUTTA

(*Received 14 June, 1982*)

Abstract : A nonempty subset A of a semigroup S is called a left pseudo-ideal if i) $ab \in A$ whenever $a, b \in A$ and $x^2 A \subseteq A$ for every element $x \in S$. The right pseudo-ideal and the pseudo-ideal are defined analogously. In (Sen 1975) we have seen that a group may contain pseudo-ideals. Now the purpose of this paper is to study the properties of pseudo-ideals in some finite groups viz. symmetric group, quaternion group, and in dihedral group. We have obtained the following results :

1) In a dihedral group all the normal subgroups, except {1}, are pseudo-ideals.

2) In a quaternion group all the subgroups, except {1}, are pseudo-ideals.

Also we have studied the properties of pseudo-ideals in finitely generated abelian groups and have determined the number of pseudo-ideals in some finite groups of order upto fifteen.

Introduction 1 : Let S be a semi group and T be a subsemigroup of S. A non-empty subset A of S is called a left T-ideal if $TA \subseteq A$ (Wallace 1962). The right T-ideal is defined analogously. A non-empty subset A of S is called a T-ideal if it is both left T-ideal and right T-ideal. In (Sen 1975) M. K. Sen has defined pseudo-ideal in a semigroup as follows : A nonempty subset A of a semigroup S is called a left pseudo-ideal if i) $ab \in A$ whenever $a, b \in A$ ii) $x^2 A \subseteq A$ for every element $x \in S$. The right pseudo-ideal and the pseudo-ideal are defined analogously. It is to be noted that in a commutative semigroup S a pseudo-ideal A may be taken as a T-ideal with $T=\bar{S}=\{x^2/x \in S\}$ when A is a subsemigroup of S ; but in a noncommutative semi-group these two notions are not the same.

In (Sen 1975), (Sen and Dutta 1979) the present author and Sen have studied the properties of the pseudo-ideals in semigroups and in groups. Now the purpose of this paper is to study the pseudo-ideals in some finite groups viz. symmetric group, quaternion group, and in dihedral group and to determine the number of pseudoideals in finitely generated abelian groups and in some finite groups.

Section—2 We recall the following properties of pseudo-ideals in groups.

Proposition 2.1 : *In a group a left pseudo-ideals is a right pseudo-ideal and vice-versa* (p 111, Sen 1975).

Proposition 2.2 : *Every pseudo-ideal of a group is also a normal subgroup but the converse is not true* (Sen 1975).

Proposition 2.3 : *A normal subgroup H of a group G will be a pseudo-ideal if and only if the factor group G/H is of exponent two* (Sen and Dutta 1979).

Proposition 2.4 : *Every finite cyclic group of even order contains one and only one proper pseudo-ideal* (Sen and Dutta 1979).

Proposition 2.5 : *A finite cyclic group of odd order does not contain any proper pseudo-ideal* (Sen and Dutta 1979).

We now prove the following :

Proposition 2.6 : *In the symmetric group S_n, A_n is the only proper pseudo-ideal for all n.*

Proof : It is known that A_n is a normal subgroup of S_n and the factor group S_n/A_n is of exponent two (*TH.* 6.8 (Hungerford)). Hence by proposition 2.3, A_n is a proper pseudo-ideal of S_n. For $n \neq 4$, A_n is the only proper normal subgroup of S_n ; hence by proposition 2.2 it follows that A_n is the only proper pseudo-ideal of S_n. Now for $n=4$, S_4 has the two proper normal subgroups viz. A_4 and V where V is the four group consisting of 1, (1,2) (3,4), (1,3) (2,4) and (1,4) (2,3). Now since $S_4/V \cong S_3$, it follows by Proposition 2.3 that V cannot be a pseudo-ideal of S_4. Hence, for all n, A_n is the only proper pseudo-ideal of S_n.

Definition 2.7 : *A semigroup S is called pseudo-simple if it does not contain any proper pseudo-ideal.*

Proposition 2.8 *The alternating group A_n is pseudo-simple for all n.*

Proof : For $n \neq 4$, A_n does not contain any proper normal subgroup (*TH* 6.10 (Hungerford)). Hence A_n does not contain any proper pseudo-ideal since a pseudo-ideal is always a normal subgroup. Hence A_n is pseudo-simple for all $n \neq 4$. For $n=4$, A_4 has a normal subgroup V consisting of 1, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3) and $A_4/V \cong Z_3$ where Z_3 is the set of integers modulo 3. Hence it follows, by proposition 2.3 that V cannot be a pseudo-ideal of A_4. Next, let N be a normal subgroup of A_4 besides V. Then N must contain at least one 3 cycle. Then by lemma 6.12 (Hungerford), it follows that $N=A_4$. So A_4 does not contain any proper normal subgroup. Hence by proposition 2.2 it follows that A_4 does not contain any proper pseudo-ideal. Hence the proposition.

Proposition 2.9 : *For each positive odd integer $n \geqslant 3$, the dihedral group D_n contains one and only one proper pseudo-ideal.*

Proof : We know that for each $n \geqslant 3$ the dihedral group D_n is a group of order $2n$ whose generators a, b satisfy i) $a^n=1, b^2=1$; $a^k \neq 1$ if $0 < k < n$; ii) $ba=a^{-1}b$ (*Th*

6.13 [Hungerford]). Then $D_n=\{1, a, \ldots, a^{n-1}, \ b, ab, \ldots , a^{n-1}b\}$ also $a^ib.\, a^jb=a^ib.\, a.\, a^{j-1}b =a^i.\, a^{-1}b.\, a^{j-1}b=a^{i-1}b\ a^{j-1}b=a^{i-j}b^2=a^{i-j}$ for $0 \leqslant i, j \leqslant n-1$. Putting $i=j$ it follows that order of the elements a^ib for $i=0\ldots, n-1$ is two. Hence it follows directly from the definition of a pseudo-ideal that the subgroup $I=\{1, a\ldots, a^{n-1}\}$ is a proper pseudo-ideal of D_n. We claim that I is the only proper pseudo-ideal of D_n. If possible, let there be another proper pseudo-ideal I_1 in D_n. Then $I \cap I_1$, which is nonempty as $1 \in I \cap I_1$ is a pseudo-id al of I; but since I is a cyclic group of odd order it follows, by proposition 2.5 that $I \cap I_1=I$. Hence $I \subseteq I_1$. Now if $I \neq I_1$, let $a^ib \in I_1$ for some i where $i=0,\ldots, n-1$. Since I_1 is a subgroup of D_n, $a^ib \in I_1$ and $a \in I_1$ imply that $I_1=D_n$ which is a contradiction. So I is the only proper pseudo-ideal of D_n. Hence the proposition.

Proposition 2.10 : *For each positive even integer* $n \geqslant 4$, D_n *has only four proper pseudo-ideals.*

Proof: It can be easily verified that $I_1=\{1, a, a^2, \ldots , a^{n-1}\}$, $I_2=\{1, a^2, a^4, \ldots , a^{n-2}\}$, $I_3=\{1, a^2, a^4, \ldots , a^{n-2}, b, a^2b, \ldots a^{n-2}b\}$ and $I_4=\{1, a^2, a^4, \ldots , a^{n-2}, ab, a^3b, \ldots , a^{n-1}b\}$ are proper pseudo-ideals of D_n. Let I be another proper pseudo-ideal of D_n. Now $I \cap I_1$, which is nonempty, is a pseudo-ideal of I_1. Since I_1 is a cyclic group of even order it has only one proper pseudo-ideal viz. I_2 [proposition 2.4]. Hence, either $I \cap I_1=I_1$ or $I \cap I_1=I_2$. If $I \cap I_1=I_1$ then $I_1 \subseteq I$. Now if $I_1 \neq I$ then some $a^ib \in I$ for $i=0, \ldots , n-1$. Now $a^ib \in I$ and $a \in I$ imply that $I=D_n$ which is a contradiction. Hence $I=I_1$. Next, if $I \cap I_1=I_2$ then $I_2 \subseteq I$. If $I_2 \neq I$ then some $a^ib \in I$ for $i=0, 1, \ldots , n-1$. As before we can show that if i is odd I coincides with I_4 and if i is even then I coincides with I_3. Thus, in any case, I is one of the four proper pseudo-ideals I_1, I_2, I_3, I_4. Hence the proposition.

Proposition 2.11. *In the dihedral group* D_n *all the normal subgroups, except* $\{ 1 \}$, *are pseudo-ideals.*

Proof: Let N be a normal subgroup $\neq \{ 1 \}$ of D_n. We shall first show that $a^i \in N$ for some $i=1, \ldots , n-1$. If possible, let none of the a^i's belongs to N; then some $a^ib \in N$ for $i=1, \ldots , n-1$. Now since N is a normal subgroup of D_n, $a^{-i}.\, a^ib.\, a^i \in N$ i.e. $ba^i \in N$. Hence $a^ib.\, ba^i \in N$ or $a^{2i} \in N$ which contradicts our assumption. Hence $a^i \in N$ for $i=1, \ldots , n-1$. Now if i is even then $\{1, a^2, \ldots , a^{n-2}\} \subseteq N$. Now $\bar{D}_n=\{x^2 / x \in D_n\}= \{1, a^2, \ldots , a^{n-2}\}$. Hence $\bar{D}_n N \subseteq N$. So N is a pseudo-ideal of D_n. If i is odd then $\{1, a, \ldots , a^{n-1}\} \subseteq N$. In this case also $\bar{D}_n N \subseteq N$. So N is a pseudo-ideal of D_n. Hence the proposition.

Proposition 2.12. *The generalised quaternion group* Q_2n *contains only four proper pseudo-ideals.*

Proof: The generalised quaternion group Q_2n is a group of order 2^n whose generators a, b satisfy i) $a^{2^{n-1}}=1$ ii) $a^{2^{n-2}}=b^2$, $ba=a^{-1} b$. Then $Q_2n=\{1, a, a^2, \ldots , a^{2^{n-1}-1},$

$b, ab, \dots, a^{2^{n-1}-1}b\}$. Now in Q_2n, $a^i b \cdot a^j b = a^{i-j} b^2 = a^{i-j+2^{n-2}}$ and $(a^i b)^2 = a^{2^{n-2}} = (a^{2^{n-3}})^2$ for $i, j = 0, 1, 2, \dots, 2^{n-1}-1$. Now $I_1 = \{1, a, \dots, a^{2^{n-1}-1}\}$, $I_2 = \{1, a^2, a^4, \dots, a^{2^{n-2}-2}\}$, $I_3 = \{1, a^2, a^4, \dots, a^{2^{n-1}-2}, b, a^2b, \dots, a^{2^{n-1}-2}b\}$ and $I_4 = \{1, a^2, \dots, a^{2^{n-1}-2}, ab, a^3b, \dots, a^{2^{n-1}-1}b\}$ are the proper pseudo-ideals of Q_2n and these are the four proper pseudo-ideals of Q_2n. If possible, let I be another proper pseudo-ideal of Q_2n, then $I \cap I_1$ is a pseudo-ideal of I_1. But I_1 is a cyclic group of even order, so it has only one proper pseudo-ideal I_2. Hence either $I \cap I_1 = I_1$ or $I \cap I_1 = I_2$ i.e. either $I_1 \subseteq I$ or $I_2 \subseteq I$. If $I_1 \neq I$ then $I = Q_2n$ which is a contradiction and if $I_2 \subseteq I$ then I coincides with one of three pseudo-ideals I_2, I_3, I_4. Hence I_1, I_2, I_3, I_4 are the only proper pseudo-ideals of Q_2n.

Proposition 2.13. *In a quaternion group all the subgroups, except* $\{1\}$, *are pseudo-ideals.*

Proof. A quaternion group Q is a group of order 8 whose generators a, b satisfy i) $a^4 = 1$ ii) $a^2 = b^2$, $ba = a^{-1}b$. Then $Q = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Let $A \neq \{1\}$ be a subgroup of Q. Then $a^2 \in A$ for if one of the a, a^2, a^3 belongs to A then $a^2 \in A$ otherwise some $a^i b$ belongs to A for $i = 0, 1, 2, 3$ and then $(a^i b)^2 = b^2 = a^2 \in A$. Consequently $\{1, a^2\} \subseteq A$. Now $\bar{Q} = \{1, a^2\}$. Hence $\bar{Q}\,A \subseteq A$ proving thereby that A is a pseudo-ideal of Q.

Hence the proposition.

Proposition 2.14. *Let a group T be generated by two elements a, b such that $a^6 = 1$, $a^k \neq 1$ if $0 < k < 6$, $a^3 = b^2$ and $ba = a^{-1}b$; then T contains only one proper pseudo ideal.*

Proof: $T = \{1, a, a^2, \dots, a^5, b, ab, \dots, a^5b\}$. Let $I_1 = \{1, a, \dots, a^5\}$; then I_1 is a proper pseudo-ideal of T. Let I be another proper pseudo-ideal of T; then $I \cap I_1$ is a pseudo-ideal of I_1. Since I_1 is a cyclic group of even order it has only one proper pseudo-ideal viz. $I_2 = \{1, a^2, a^4\}$. Hence either $I \cap I_1 = I_1$ or $I \cap I_1 = I_2$. If $I \cap I_1 = I_2$ then $I_2 \subseteq I$. Now since I is a pseudo-ideal $b^2 I_2 \subseteq I$ or $a^3 I_2 \subseteq I$ i.e. $\{a^3, a^5, a\} \subseteq I$. Hence $\{1, a, \dots, a^5\} \subseteq I$. If $\{1, a, \dots, a^5\} \neq I$ then some $a^i b \in I$ for $i = 0, 1, \dots, 5$. Now $a^i b \in I$ and $a \in I$ imply $I = T$ which is a contradiction. Hence $I = I_1$. Similarly when $I \cap I_1 = I_1$ we can show that $I = I_1$. Hence the proposition.

Proposition 2.15. *Let G be a periodic group in which every element is of odd order ; then G is pseudo-simple* (2.15).

Proof: Let I be a pseudo-ideal of G. Since every pseudo-ideal is a normal subgroup, $1 \in I$. Let a be an element of G and its order be $2k+1$ where k is a positive integer. Then $a^{2k+1} = 1 \in I$. Now $a^{-k} \in G$ and $a = (a^{-k})^2 \cdot a^{2k+1} \in I$ as I is a pseudo-ideal of G. So $I = G$. Hence there does not exist any proper pseudo-ideal in G. In other words G is pseudo simple.

Corollary : *Every group of odd order is pseudo-simple.*

Section 3 **:** In this section we have determined when a finitely generated abelian group will be pseudo-simple and the number of proper pseudo-ideals in a finitely generated abelian group in case it is not pseudo-simple.

Definition 3.1. *A p-group (where p is any prime number) is a group in which the order of each element is some power of p.*

Lemma 3.2. *Let* G_1 *and* G_2 *be two groups and* I_1 *and* I_2 *be two pseudo-ideals of* G_1 *and* G_2 *respectively ; then* $I_1 \oplus I_2$ *is a pseudo-ideal of* $G_1 \oplus G_2$ *where* $G_1 \oplus G_2$ *denotes the direct sum of* G_1 *and* G_2 *and* $I_1 \oplus I_2$ *will be a proper pseudo-ideal if one of the* I_1, I_2 *is proper.*

Lemma 3.3. *If* $G_1 \oplus G_2$ *contains a proper pseudo-ideal, then there exists a proper pseudo-ideal either in* G_1 *or in* G_2 *or in both* G_1 *and* G_2, *where* G_1 *and* G_2 *are two groups.*

Proposition 3.4. *A finite abelian p-group G will be pseudo-simple if and only if* $p \neq 2$ *and when* $p=2$ *the number of pseudo-ideals of G will be at least* 2^r-1 *where r is the type* [Macdonald 1968] *of G.*

Proof: Let the finite abelian p-group G be pseudo-simple. If possible, let $p=2$ Now G is isomorphic to the direct sum of cyclic p-group (Th 5.05 [Macdonald 1968]). Each of the direct summand being a finite abelian p-group has p-power order (P 84 [Macdonald 1968]). So each of the direct summand is a cyclic p-group of even order since $p=2$. Consequently, each of the direct summand contains only one proper pseudo-ideal, by proposition 2.4. Hence, it follows by lemma 3.2 that the group G which is isomorphic to the direct sum of cyclic p-groups contains proper pseudo-ideals which contradicts our assumption. Hence $p \neq 2$. Conversely, suppose that $p \neq 2$; then the order of the each of the elements of G is odd. Hence by proposition 2.15, it follows that G is pseudo-simple. Now if the type of the group G be r ; then G is isomorphic to the direct sum of cyclic groups of orders $p^{n_1}, p^{n_2}, \ldots, p^{n_r}$ where $n_1, n_2, \ldots, n_r$ are positive integers. Since $p=2$, each of the direct summands contains one and only one proper pseudo-ideal. Hence, by lemma 3.2 it follows that G contains proper pseudo-ideals and the number of these proper pseudo-ideals is *at least* 2^r-1.

Proposition 3.5. *A finite abelian group G will be pseudo-simple if and only if it does not contain elements of 2-power order.*

Proof: Let the finite abelian group G contain the set G_2 of the elements of 2-power order ; then G_2 forms a subgroup of G. So G_2 is a finite abelian p-group where $p=2$

Consequently, by proposition 3.4, G_2 contains proper pseudo-ideals. Now G is the direct sum of the subgroups G_i where G_i is the set of all p_i-power order elements of G (Th 5.01 [Kochendorffer]). Now the direct summand G_2 contains proper pseudo-ideals, hence G contains proper pseudo-ideals. Conversely let G does not contain elements of 2-power order. Then all of the elements of G are of odd order. Hence, by proposition 2.15 G is pseudo-simple.

Definition 3.6. *The rank of a finitely generated abelian group is the number of infinite direct summands in any representation of the group as the direct sum of cyclic groups* (P 94 (Macdonald 1968)).

Proposition 3.7. *The rank of a finitely generated abelian group G is equal to the rank of any pseudo-ideal of G.*

Proof: Let B be a pseudo-ideal of G. Now since B is also normal subgroup of G, it follows that rank of G is equal to the sum of the ranks of B and A/B. Now A/B being a finitely generated abelian periodic group, is finite. Hence its rank is equal to zero. Hence the proposition.

Proposition 3.8. *A torsion free finitely generated abelian group G which is non-trivial of rank n contains at least 2^n-1 proper pseudo-ideals.*

Proof: Since the torsion free finitely generated abelian group G is of rank n, G is isomorphic to the direct sum of a set of n infinite cyclic groups (Th 5.09 (Macdonald 1968)). Now an infinite cyclic group being isomorphic to the additive group of all integers contains only one proper pseudo-ideal. Hence the group G contains *at least* 2^n-1 proper pseudo-ideals.

Proposition 3.9. *A finitely generated abelian group contains a finite number of proper pseudo-ideals if its rank is not equal to zero.*

Proof: By theorem 2.1 of [Hungerford] it follows that $G \cong Z_{m_1} \oplus Z_{m_2} \oplus \cdots \oplus Z_{m_t} \oplus (Z \oplus \cdots \oplus Z)$ where $m_1 > 1$, $m_1 \mid m^2 \mid \cdots \mid m_t$, $Z \oplus \cdots \oplus Z$ has rank $\neq 0$ and Z is the additive group of all integers. Since the infinite cyclic group Z has a proper pseudo-ideal, G has also a proper pseudo-ideal. Also the number of proper pseudo-ideals of G is finite as each of the direct summands in the above representation contains a finite number of proper pseudo-ideals.

Concluding remark

Lastly, we list the number of proper pseudo-ideals in the groups of order upto fifteen.

Order	Distinct Groups	Number of pseudo-ideals
2	Z_2	1
3	Z_3	0
4	$Z_2 \oplus Z_2$, Z_4	>3, 1
5	Z_5	0
6	Z_6, D_3	1
7	Z_7	0
8	$Z_2 \oplus Z_2 \oplus Z_2$, $Z_2 \oplus Z_4$, Z_8	>7, >3, 1
	Q_8, D_4	4, 4.
9	$Z_3 \oplus Z_3$, Z_9	0
10	Z_{10}, D_5	1
11	Z_{11}	0
12	$Z_2 \oplus Z_6$, Z_{12}, A_4, D_6, T	>3, 1, 0, 1, 1
13	Z_{13}	0
14	Z_{14}, D_7	1
15	Z_{15}	0.

Acknowledgement : The author is indebted to Dr. M. K. Sen for several useful discussions concerning this problem.

References

Hungerford, T. W. (——) : Algebra, Holt, Rinehart and Winston, Inc., New York.

Kochendorffer (——) : Group theory, McGraw-Hill, London.

Macdonald, I. D. (1968) : The Theory of groups. Oxford Univ. Press.

Sen, M. K. (1975) : On pseudo-ideals of semigroups. *Bull. Cal. Math. Soc.*, 67, 109.

Sen, M. K. and Dutta, T. K. (1979) : On pseudo-ideals of groups. Analele Stiinti fice ale universitatii Al. I. *CUZA " din Iasi.*, XXV, S.I.a, f.z.

Wallace, A. D. (1962) : Relative ideals in semigroups I colloq. Math., 9, 55.

Wallace, A. D. (1963) : Relative ideals in semigroups II *Acta. Math. Acad. Sci. Hunger*, 14, 137.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA
35, BALLYGUNGE CIRCULAR ROAD
CALCUTTA-700 019
INDIA.

Bull. Cal. Math. Soc., 75, 339.352 (1983)

ON SOME MAGNETOHYDROSTATIC CONFIGURATIONS IN PARABOLIC COORDINATES

Pramod Kumar Mittal

(*Received 2 August 1978 : Revision Received 2 October 1978*)

Summary: In the present paper an attempt has been made to find some magnetic fields with conservative Lorentz force in parabolic co-ordinates, neglecting the effects of displacement current. Some self-superposable hydrodynamic flows have also been observed. It has further been shown that the self-superposable flow of an electrically conducting incompressible fluid permeated by a magnetic field with conservative Lorentz force may constitute a magnetohydrostatic configuration under certain conditions. Eight such possible configurations have been attempted and the distribution of pressure for each configuration has been found.

Introduction : In hydromagnetics the Lorentz force plays many important roles. Various authors (Bhattacharya & Banga 1977, Gurtu 1973 & Mittal) have tried to represent it by various co-ordinate systems. In this paper author has made an attempt to find some magnetic fields in parabolic co-ordinates for which the Lorentz force is conservative. For this purpose some magnetic fields whose Lorentz force can be represented by the gradient of a scalar quantity have been determined. As such these magnetic fields will have conservative Lorentz force. In writing the equation of the problem all basic assumptions in the theory of hydromagnetics, including the assumption that 'displacement current field being small compared to the prevalent electric field can be discarded', have been taken into account.

It was shown by Ballabh (1940) that when a flow with velocity $\bar{q}$ satisfies the condition

$$\text{curl}\,(\,\bar{q}\times\text{curl}\,\bar{q}\,)=0$$

it becomes self-superposable. The fluid velocities are found for which ($\bar{q}\times$ curl $\bar{q}$) can be represented as the gradient of a scalar quantity. Obviously such flows will be self-superposable.

The magnetohydrostatic configurations may develop when the magnetic fields with conservative Lorentz force will act upon self-superposable flow of electrically conducting incompressible fluids. The author has attempted to find out the conditions and the pressure distributions for such possible configurations.

Lorentz force as the Gradient OF a scalar quantity

Let the Lorentz force be represented by

$$\bar{L} = -\frac{\mu}{4\pi}(\bar{H} \times \text{curl } \bar{H}) \qquad \dots \quad (1)$$

$\bar{H}$ and μ denote the magnetic field and permeability respectively. Now, suppose that ϕ be the required scalar so that we may express $\bar{L}$ by the equation

$$\bar{L} = -\frac{\mu}{4\pi}\,\text{grad}\,\phi \qquad \dots \quad (2)$$

Equating the coefficients of various terms of equation (1) we get

$$L_u = 2H_v\left[\sqrt{\frac{u}{u+v}}\frac{\partial}{\partial u}(\sqrt{u+v}\,H_v) - \sqrt{\frac{v}{u+v}}\frac{\partial}{\partial v}(\sqrt{u+v}\,H_u)\right]$$

$$- H_w\left[\frac{1}{\sqrt{uv}}\frac{\partial H_u}{\partial w} - \frac{2}{\sqrt{u+v}}\frac{\partial}{\partial u}(\sqrt{u}\,H_w)\right] \qquad \dots \quad (3)$$

$$L_v = H_w\left[\frac{2}{\sqrt{u+v}}\frac{2}{\partial v}(\sqrt{v}\,H_w) - \frac{1}{\sqrt{uv}}\frac{\partial H_{v}}{\partial w}\right]$$

$$- 2H_u\left[\sqrt{\frac{u}{u+v}}\frac{\partial}{\partial u}(\sqrt{u+v}\,H_v) - \sqrt{\frac{v}{u+v}}\frac{\partial}{\partial v}(\sqrt{u+v}\,H_u)\right] \qquad \dots \quad (4)$$

$$L_w = H_u\left[\frac{1}{\sqrt{uv}}\frac{\partial H_u}{\partial w} - \frac{2}{\sqrt{u+v}}\frac{\partial}{\partial u}(\sqrt{u}\,H_w)\right]$$

$$- H_v\left[\frac{2}{\sqrt{u+v}}\frac{\partial}{\partial v}(\sqrt{v}\,H_w) - \frac{1}{\sqrt{uv}}\frac{\partial H_v}{\partial w}\right] \qquad \dots \quad (5)$$

where (L_u, L_v, L_w) and (H_u, H_v, H_w) are the components of $\bar{L}$ and $\bar{H}$ at any point (u, v, w) in parabolic co-ordinates. Also,

$$\text{grad}\,\phi = 2\sqrt{\frac{u}{u+v}}\frac{\partial\phi}{\partial u}\,i_1 + 2\sqrt{\frac{v}{u+v}}\frac{\partial\phi}{\partial v}\,i_2 + \frac{1}{\sqrt{uv}}\frac{\partial\phi}{\partial w}\,i_3 \qquad \dots \quad (6)$$

Here i_1, i_2 and i_3 are the unit vectors along u-, v- and w- axes respectively. From eqn. (2) with the help of eqns. (3), (4), (5) and (6) we get

$$\frac{\partial\phi}{\partial u} = \frac{1}{2}\sqrt{\frac{u+v}{u}}\,L_u \qquad \dots \quad (7)$$

$$\frac{\partial\phi}{\partial v} = \frac{1}{2}\sqrt{\frac{u+v}{v}}\,L_v \qquad \dots \quad (8)$$

$$\frac{\partial\phi}{\partial w} = \sqrt{uv}\, L_w \quad \ldots \quad (9)$$

values of $\frac{\partial\phi}{\partial u}$, $\frac{\partial\phi}{\partial v}$ and $\frac{\partial\phi}{\partial w}$ when substituted in

$$d\phi = \left(\frac{\partial\phi}{\partial u}\, du + \frac{\partial\phi}{\partial v}\, dv + \frac{\partial\phi}{\partial w}\, dw \right)$$

will give ϕ in the form

$$\phi = \int \left(\frac{\partial\phi}{\partial\phi}\, du + \frac{\partial\phi}{\partial v}\, dv + \frac{\partial\phi}{\partial w}\, dw \right) \quad \ldots \quad (10)$$

The magnetic field should also satisfy the equation

$$\operatorname{div} \bar{H} = 0$$

which in parabolic co-ordinates becomes

$$\frac{1}{u+v}\left[\frac{\partial}{\partial_u}\left\{ \sqrt{u(u+v)}\, H_u \right\} + \frac{\partial}{\partial v}\left\{ \sqrt{v(u+v)}\, H_v \right\} + \frac{1}{\sqrt{uv}}\, \frac{\partial H_w}{\partial w} \right] = 0 \quad \ldots \quad (11)$$

In order to make the equation (10) integrable we may consider following cases—

Case I. When $H_u = 0$.

(i) Let

$$H_v = \frac{1}{2}\sqrt{\frac{(u+v)^3}{v}}\, f_1(w)$$

where $f_1(w)$ is an integrable function of w. Substituting the value of H_u and H_v in eqn. (11) we get

$$H_w = -\sqrt{uv}\; F_1(w)$$

where

$$F_1(w) = \int f_1(w)\, dw \quad \ldots \quad (12)$$

Thus a magnetic field which satisfies eqn. (12) is

$$\left.\begin{aligned} H_u &= 0 \\ H_v &= \frac{1}{2}\sqrt{\frac{(u+v)^3}{v}}\, f_1(w) \\ H_w &= -\sqrt{uv}\, F_1(w) \end{aligned}\right\} \quad \ldots \quad (13)$$

By substituting the values of H_u, H_v and H_w from eqn. (13) in eqns. (3), (4) and (5) we get

$$L_u = \sqrt{u}\,\frac{(u+v)^3}{v}\,\{f_1(w)\}^2 + 2v\sqrt{\frac{u}{u+v}}\{F_1(w)\}^2 \qquad \dots \quad (14)$$

$$L_v = \left[2u\sqrt{\frac{v}{u+v}}\,F_1(w) + \tfrac{1}{2}f'_1(w)\sqrt{\frac{u+v}{v}}\right]F_1(w) \qquad \dots \quad (15)$$

$$L_w = \left[(u+v)\sqrt{\frac{u}{v}}\{F_1(w)\,f_1(w)\} + \frac{(u+v)^3}{4v}\sqrt{\frac{1}{uv}}\{f_1(w)\,f'_1(w)\}\right] \qquad \dots \quad (16)$$

These values of L_u, L_v, L_w give

$$\phi = \Big[u(u+3v)\{F_1(w)\}^2 + \frac{(u+v)^3}{28\,v}(4\sqrt{u+v}+7)\{f_1(w)\}^2$$

$$+\tfrac{1}{4}\{f'_1(w)\,F_1(w)\}(u^2\log v + 2uv + \tfrac{1}{2}v^2) \qquad \dots \quad (17)$$

Now, it is clear that the magnetic field (13) must have conservative Lorentz force as it has been represented by the gradient of a scalar quantity ϕ given by eqn. (17).

(ii) One more solution to satisfy the eqn. (11) is possible where—

$$\left.\begin{aligned} H_u &= 0 \\ H_v &= \frac{A}{\sqrt{v(u+v)}} \\ H_w &= C \end{aligned}\right\} \qquad \dots \quad (18)$$

Here A and C are constants.

For the magnetic field represented by eqn. (18) it may readily be shown that the Lorentz force can be represented by the gradient of a scalar quantity ϕ given by

$$\phi = D\log uv - \frac{Ew}{(u+v)}\sqrt{\frac{u}{v}} \qquad \dots \quad (19)$$

where $D = \frac{1}{2}C^2$ and $E = AC$.

No doubt, the magnetic field denoted by eqn. (18) will have conservative Lorentz force.

(iii) Eqn. (11) may also be satisfied by—

$$\left.\begin{aligned} H_u &= 0 \\ H_v &= \frac{A_1}{\sqrt{v(n+v)}} \\ H_w &= 0 \end{aligned}\right\} \qquad \dots \quad (20)$$

for which

and

$$\left.\begin{aligned} \bar{L} &= 0 \\ \phi &= \text{constant} \end{aligned}\right\} \qquad \dots \quad (21)$$

Since, for the magnetic field (21) Lorentz force is zero hence it will be a force-free field.

Case II. When $H_v = 0$.

(i) Let

$$H_u = \tfrac{1}{2} B f_2(w) \sqrt{\frac{(u+v)^3}{u}}$$

where $f_2(w)$ is an integrable function of w and B is an arbitrary constant. By substituting the values of H_u and H_v in eqn. (11) we get—

$$H_w = -B \sqrt{uv}\, F_2(w)$$

where $$F_2(w) = \int f_2(w)\, dw \qquad \dots \quad (22)$$

Then a magnetic field which satisfies eqn. (11) is

$$\left.\begin{aligned} H_u &= \tfrac{1}{2} B f_2(w) \sqrt{\frac{(u+v)^3}{u}} \\ H_v &= 0 \\ H_w &= -B \sqrt{uv}\, F_2(w) \end{aligned}\right\} \qquad \dots \quad (23)$$

For this magnetic field we will have

$$L_u = B^2 F_2(w) \left[\frac{1}{2} \sqrt{\frac{(u+v)^3}{u}}\, f'_2(w) + 2F_2(w)\, u \sqrt{\frac{v}{u+v}} \right] \qquad \dots \quad (24)$$

$$L_v = B^2 \left[2u \sqrt{\frac{v}{u+v}} \{F_2(w)\}^2 + \sqrt{v}\, \frac{(u+v)^3}{u} \{f_2(w)\}^2 \right] \qquad \dots \quad (25)$$

$$L_w = \tfrac{1}{2} B^2 \left[\frac{(u+v)^3}{2u\sqrt{uv}} \{ f_2(w) f'_2(w) \} + 2(u+v) \sqrt{\frac{v}{u}}\ \{ F_2(w) f_2(w) \} \right] \qquad \dots \quad (26)$$

and

$$\phi = \left[\tfrac{1}{2} B^2 v(u+v) \{ F_2(w) \}^2 + \frac{B^2(u+v)^3}{u} \left\{ \frac{\sqrt{u+v}}{7} + \frac{1}{8} \right\} \{ f_2(w) \}^2 \right.$$
$$\left. + \frac{1}{8} B^2 (u^2 + 4uv + 2v^2 \log u) \{ f'_2(w) F_2(w) \} \right] \qquad \dots \quad (27)$$

Thus for the magnetic field given by eqn. (23) the Lorentz force is conservative.

(ii) Eqn. (11) may also be satisfied by

$$\left.\begin{aligned} H_u &= \frac{B_1}{\sqrt{u(u+v)}} \\ H_v &= 0 \\ H_w &= D \end{aligned}\right\} \qquad \cdots \quad (28)$$

Here B_1 and D are the constants.

For the magnetic field represented by eqn. (28) it may readily be shown that the Lorentz force can be represented by the gradient of a scalar quantity ϕ given by

$$\phi = F \log uv - \frac{Gw}{u+v}\sqrt{\frac{v}{u}} \qquad \cdots \quad (29)$$

where $F = \frac{1}{2}D^2$, $G = DB_1$.

Thus magnetic field given by eqn. (28) must have conservative Lorentz force.

(iii) One more possible solution of eqn. (11) may be—

$$\left.\begin{aligned} H_u &= \frac{B_2}{\sqrt{u(u+v)}} \\ H_v &= 0 \\ H_w &= 0 \end{aligned}\right\} \qquad \cdots \quad (30)$$

In this case we can show that—

$$\text{and} \quad \left.\begin{aligned} \bar{L} &= 0 \\ \phi &= \text{constant} \end{aligned}\right\} \qquad \cdots \quad (31)$$

Hence the magnetic field represented by eqn. (30) will be force-free.

Case III. When $H_w = 0$

(i) Let $H_u = P\sqrt{\dfrac{u}{u+v}}\; f_3(w)$

where $f_3(w)$ is a function of w and P is a constant.

By substituting the values of H_u and H_w in eqn. (11) we get—

$$H_v = -P\sqrt{\frac{v}{u+v}}\; f_3(w)$$

Thus a magnetic field which satisfies eqn. (11) is—

$$\left.\begin{aligned} H_u &= P\sqrt{\frac{u}{u+v}}\ \ f_8(w) \\ H_v &= -P\sqrt{\frac{v}{u+v}}\ \ f_8(w) \\ H_w &= 0 \end{aligned}\right\} \quad \dots \quad (32)$$

For this magnetic field we can get

$$\left.\begin{aligned} L_u &= L_v = 0 \\ L_w &= \frac{P^2\,\{f_8(w)\,f'_8(w)\}}{\sqrt{uv}} \end{aligned}\right. \quad \dots \quad (33)$$

and

$$\phi = P^2\,\{f_8(w)\}^2 \quad \dots \quad (34)$$

Now, it is clear that the magnetic field (32) must have conservative Lorentz force as it has been represented by the gradient of a scalar quantity ϕ given by eqn. (34).

(ii) One more solution of eqn. (11) may be possible, which is

$$\left.\begin{aligned} H_u &= P_1\sqrt{\frac{u}{u+v}} \\ H_v &= P_1\sqrt{\frac{v}{u+v}} \\ H_w &= 0 \end{aligned}\right\} \quad \dots \quad (35)$$

For this magnetic field we will have

$$L_u = L_v = L_w = 0 \quad \dots \quad (36)$$

which gives

and

$$\left.\begin{aligned} \bar{L} &= 0 \\ \phi &= \text{constant.} \end{aligned}\right\} \quad \dots \quad (37)$$

Obviously, the magnetic field denoted by eqn. (35) will be force-free.

In the foregoing discussions all the eight magnetic fields in parabolic coordinates given by eqns. (13), (18), (20), (23), (28), (30), (32) and (35) have conservative Lorentz force. Out of these the magnetic fields given by eqns. (13), (18), (23), (28) and (32) are pressure-balanced fields and it is a well known fact that in such fields the lines of force and current are situated in the surface ϕ=constant, The other three fields are force-free magnetic fields.

Some Self-Superposable Incompressible Hydrodynamic Flows

It was shown by Ballabh (1940) that a fluid motion $\bar{q}$ will be self-superposable if

$$\text{curl}\,(\bar{q}\times\text{curl}\,\bar{q})=0 \qquad \cdots \quad (38)$$

For the incompressible fluid we also have

$$\text{div}\;\bar{q}=0 \qquad \cdots \quad (39)$$

Let

$$\bar{q}\times\text{curl}\,\bar{q}=\bar{P} \qquad \cdots \quad (40)$$

Now, if we consider the flow of an incompressible fluid for which $\bar{P}$ can be represented as the gradient of a scalar quantity θ (say), *i.e.*,

$$\bar{P}=\text{grad}\;\theta \qquad \cdots \quad (41)$$

then by operating 'curl' on both the sides we will have

$$\text{curl}\;\bar{P}=\text{curl grad}\;\theta=0$$

or,

$$\text{curl}\,(\bar{q}\times\text{curl}\,\bar{q})=0$$

which is same as the condition (38). Thus a flow of an incompressible fluid for which we can represent $\bar{P}$ as the gradient of a scalar, will be self-superposable.

By the process adopted in the preceding pages and in accordance with the eqns. (13), (18), (20), (23), (28), (30), (32) and (35) we may write some incompressible flows in parabolic co-ordinates for which $\bar{P}$ can be represented by a scalar quantity as follows—

(i)
$$\left.\begin{aligned} q_u&=0\\ q_v&=\tfrac{1}{2}\sqrt{\frac{(u+v)^3}{v}}\quad L_1(w)\\ q_w&=-\sqrt{uv}\,M_1(w)\end{aligned}\right\} \qquad \cdots \quad (42)$$

(ii)
$$\left.\begin{aligned} q_u&=0\\ q_v&=\frac{A'}{\sqrt{v(u+v)}}\\ q_w&=C'\end{aligned}\right\} \qquad \cdots \quad (43)$$

(iii)
$$\left.\begin{aligned} q_u &= 0 \\ q_v &= \frac{A'_1}{\sqrt{v(u+v)}} \\ q_w &= 0 \end{aligned}\right\} \qquad \ldots \quad (44)$$

(iv)
$$\left.\begin{aligned} q_u &= \tfrac{1}{2}B'\sqrt{\frac{(u+v)^3}{u}}\, L_2(w) \\ q_v &= 0 \\ q_w &= -B'\sqrt{uv}\; M_2(w) \end{aligned}\right\} \qquad \ldots \quad (45)$$

(v)
$$\left.\begin{aligned} q_u &= \frac{B'_2}{\sqrt{u(u+v)}} \\ q_v &= 0 \\ q_w &= D' \end{aligned}\right\} \qquad \ldots \quad (46)$$

(vi)
$$\left.\begin{aligned} q_u &= \frac{B'_2}{\sqrt{u(u+v)}} \\ q_v &= 0 \\ q_w &= 0 \end{aligned}\right\} \qquad \ldots \quad (47)$$

(vii)
$$\left.\begin{aligned} q_u &= P'\sqrt{\frac{u}{u+v}}\; L_3(w) \\ q_v &= P'\sqrt{\frac{v}{u+v}}\; L_3(w) \\ q_w &= 0 \end{aligned}\right\} \qquad \ldots \quad (48)$$

(viii)
$$\left.\begin{aligned} q_u &= P'_1\sqrt{\frac{u}{u+v}} \\ q_v &= P'_1\sqrt{\frac{v}{u+v}} \\ q_w &= 0 \end{aligned}\right\} \qquad \ldots \quad (49)$$

In all the eight flows given above q_u, q_v, q_w are the components of fluid velocity $\bar{q}$ at any point (u, v, w) in parabolic co-ordinates. A', A'_1, B', B'_1, B'_2, P' and P'_1 are the constants. $L_n(w)$ and $M_n(w)$ $(n=1,2,3)$ are the functions of w such that $L_n(w)$ are integrable and

$$M_n(w)=\int L_n(w)\ dw \qquad \cdots \quad (50)$$

$n=1,2,3$

It should also be noted here that flows denoted by eqns. (44), (47) and (49) are Beltrami flows as for these flows $\bar{P}$ is zero.

For each of the flow given by eqns. (42)-(49) the scalar quantity θ can be determined. It will be interesting to note that this θ is nothing but Bernoulli function (Mises 1958) given by

$$\theta=\frac{q^2}{2g}+h+\frac{P}{g} \qquad \cdots \quad (51)$$

where q, g, h, and P denote velocity, acceleration due to gravity, height above some arbitrary horizontal plane of reference and the pressure head.

It is a well known fact that for an incompressible fluid the pressure head P is given by

$$P=\frac{p}{\rho_0}+\text{constant} \qquad \cdots \quad (52)$$

where p is pressure distribution of the flow.

Also, if the motion of the fluid be steady and slow then value of h can be taken, without much loss of generality, as u in flows (i), (ii) and (iii) ; as v in flows (iv), (v) and (vi) ; and as w in flows (vii) and (viii).

Thus for each flow the pressure distribution can be found with the help of eqns. (51) and (52).

Some Magnetohydrostatic Configurations

If $\bar{q}$ denotes the fluid velocity of an electrically conducting incompressible fluid premeated by a magnetic field $\bar{H}$ and ρ the density, the steady states consist of all solutions of the equations (Kendall 1964)—

$$\rho\,(\text{curl}\ \bar{q})\times\bar{q}-\frac{\mu}{4\pi}(\text{curl}\ \bar{H})\times\bar{H}=-\text{grad}\ p-\rho\ \text{grad}\ (\tfrac{1}{2}q^2)+\rho\ \bar{F} \qquad \cdots \quad (53)$$

$$\text{curl}\ (\bar{q}\times\bar{H})=0 \qquad \cdots \quad (54)$$

Here, the body force $\bar{F}$ will contain no frictional forces.

Now, let us consider—

(i) the flow of an electrically conducting incompressible fluid with velocity given by eqn. (42) acted upon by the magnetic field given by eqn. (13). If

$$\left.\begin{aligned} f_1(w) &= \pm\sqrt{\frac{\mu}{4\pi\rho}}\,L_1(w) \\ \text{and}\quad F_1(w) &= \pm\sqrt{\frac{\mu}{4\pi\rho}}\,M_1(w) \end{aligned}\right\} \qquad \ldots \quad (55)$$

then the flow (42) and magnetic field (13) will satisfy the eqn. (54). Also, we will have

$$\bar{q} = \pm\frac{\bar{H}\sqrt{\mu}}{\sqrt{4\pi\rho}} \qquad \ldots \quad (56)$$

Chandrasekhar (1961) has shown that eqn. (56) is always a solution of eqns. (53) and (54) if $\bar{F}=0$ (no external forces) and if

$$p+\tfrac{1}{2}\rho\, q^2 = \text{constant} \qquad \ldots \quad (57)$$

Thus the flow of an electrically conducting incompressible fluid with velocity (42) and acted upon by magnetic field given by eqn. (13) constitute a magnetohydrostatic configuration if there is no external force, eqn. (55) is satisfied and the pressure distribution is given by

$$P = \text{constant} - \frac{\mu}{8\pi}\left[\frac{1}{4}\,\frac{(u+v)^3}{v}\left\{f_1(w)\right\}^2 + uv\left\{F_1(w)\right\}^2\right] \qquad \ldots \quad (58)$$

Similarly we can show some more magnetohydrostatic configurations in parabolic co-ordinates as follows—

(ii) The flow of an electrically conducting incompressible fluid with velocity given by eqn. (43) acted upon by the magnetic field given by eqn. (18), if

$$\left.\begin{aligned} A' &= \pm\sqrt{\frac{\mu}{4\pi\rho}}\quad A \\ C &= C' \\ F &= 0 \end{aligned}\right\} \qquad \ldots \quad (59)$$

and

$$p = \text{constant} - \frac{\mu}{8\pi}\left[\frac{A^2}{v(u+v)} + C^2\right] \qquad \ldots \quad (60)$$

(iii) Flow given by eqn. (44) under the action of magnetic field given by eqn. (20), if—

$$\left.\begin{aligned} A_1' &= \pm\sqrt{\frac{\mu}{4\pi\rho}}\;A_1 \\ \bar{F} &= 0 \end{aligned}\right\} \quad \ldots \quad (61)$$

and

$$p=\text{constant}-\frac{\mu\,A_1^2}{8\pi v(u+v)} \quad \ldots \quad (62)$$

(iv) Flow given by eqn. (45) acted upon by the magnetic field given by eqn. (23) if

$$\left.\begin{aligned} B' &= \pm\sqrt{\frac{\mu}{4\pi\rho}}\;B \\ L_2(w) &= f_2(w) \\ M_2(w) &= F_2(w) \\ \bar{F} &= 0 \end{aligned}\right\} \quad \ldots \quad (63)$$

and

$$p=\text{constant}-\frac{\mu B^2}{8\pi}\left[\frac{1}{4}\,\frac{(u+v)^3}{u}\left\{f_2(w)\right\}^2+uv\left\{F_2(w)\right\}^2\right] \quad \ldots \quad (64)$$

(v) Flow (46) permeated by magnetic field (28) if—

$$\left.\begin{aligned} B_1' &= \pm\sqrt{\frac{\mu}{4\pi\rho}}\;B_1 \\ D' &= D \\ \bar{F} &= 0 \end{aligned}\right\} \quad \ldots \quad (65)$$

and

$$p=\text{constant}-\frac{\mu}{8\pi}\left[\frac{B_1^2}{u(u+v)}+D^2\right] \quad \ldots \quad (66)$$

(vi) Flow (47) acted by the magnetic field given by eqn. (30) when—

$$\left.\begin{aligned} B_2' &= \pm\sqrt{\frac{\mu}{4\pi\rho}}\;B_2 \\ \bar{F} &= 0 \end{aligned}\right\} \quad \ldots \quad (67)$$

and

$$p=\text{constant}-\frac{\mu\,B_2^2}{8\pi u(u+v)} \quad \ldots \quad (68)$$

(vii) Flow (48) permeated by the magnetic field represented by eqn. (32) when –

$$\left.\begin{aligned} P' &= \pm\sqrt{\frac{\mu}{4\pi\rho}}\,P \\ L_3(w) &= f_3(w) \\ \bar{F} &= 0 \end{aligned}\right\} \qquad \ldots (69)$$

and

$$p = \text{constant} - \frac{\mu P^2 \{f_3(w)\}^2}{8\pi} \qquad \ldots (70)$$

(viii) Flow (49) acted upon by the magnetic field given by eqn. (35) when—

$$\left.\begin{aligned} P_1' &= P_1 \\ \bar{F} &= 0 \end{aligned}\right\} \qquad \ldots (71)$$

and

$$p = \text{constant}. \qquad \ldots (72)$$

Out of the above eight magnetohydrostatic configurations in parabolic co-ordinates, (i), (ii), (iv), (v) and (viii) are the equipartition solutions. In these configurations the magnetic lines of force behave like elastic strings with a tension $= H^2/4\pi$ and being acted upon by the magnetic pressure $H^2/8\pi$ and the hydrostatic pressure $p=\phi/4\pi$ (Lundquist 1950). Other these configurations are that of force-free type. In these the magnetic field is every where parallel to the direction of current flow. In these cases the hydrostatic equilibrium is the same as in the absence of a magnetic field. Although in all the eight configurations no secondary flow is possible because in all of these the magnetic forces can just balance the centrifugal effects (Sherchiff 1977).

Also, in each of the case the pressure distribution, before and after the application of magnetic field, can be determined and compared. This can throw fresh light on the study of magnetohydrostatic configurations.

Acknowledgement : My respectful thanks are due to Dr. R. N. Gupta, Principal, Hindu College, Moradabad for his helpful guidance and continual encouragement. The work has financially been supported by U. G. C.

References

Ballabh, R. (1940) : *Proc. Benaras Math. Soc.* (N. S.), **II**, 69.

Bhattacharya, T. K. and Banga, P. (1977) : *The Mathematics Education*, **XI**, No. 2, 37.

Chandrasekhar, S. (1961) : Hydrodynamic and Hydromagnetic Stability, Oxford University Press.

Gurtu, V. K. (1973) : Progress of Mathematics, **1**, 38.

Hinchey, F. A. (1976) : Vectors and Tensors for Engineers and Scientists, Wiley Eastern Ltd.

Kendall, P. C. and Plumpton, C. (1964) : Magnetohydrodynamics, **I**, Pergamon Press.

Lundquist, S. (1950) : Arkiv For Fysik, Bd 2 nr 35, 361.

Mittal, P. K. ( ) : *Bull. Cal. Math. Soc.* (to appear).

Mittal, P. K. ( ) : Communicated

Mises, R. V. (1958) : Mathematical Theory of Compressible Fluid Flow, Academic Press.

Shercliff, J. A. (1977) : *J. Fluid Mech.*, **80**, pt. I, 179.

DEPARTMENT OF MATHEMATICS
GOVT. RAZA P. G. COLLEGE
RAMPUR, U. P.

Bull. Cal. Math. **75**, 353-361 (1983)

GENERATING FUNCTIONS FOR JACOBI GEGENBAUER AND BEDIENTS POLYNOMIALS

G. N. CHANDWANI

(*Received 8 November, 1978*)

Abstract—In the present paper, we have established the following relations involving Kampe de Feriet's double hypergeometric functions of superior order.

$$F\left[\begin{array}{c|c|c} p & a_1,\ldots,a_p & \\ 0 & \cdot\ \cdot\ \cdot & ut, \\ q & c_1,\ldots,c_a & vt \\ 1 & b,b' & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n (vt)^n}{\prod_{j=1}^{a}(c_j)_n (b')_n\, n\,!}\quad {}_2F_1\left[-n,\ 1-b'-n\ ;\ b\quad \frac{u}{v}\right],$$

and,

$$F\left[\begin{array}{c|c|c} p & a_1,\ldots,a_p & ut, \\ 0 & \cdot\ \cdot\ \cdot & \\ q & c_1,\ldots,c_q & \\ 2 & b,b',\ b_1,b'_1 & vt \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n (vt)^n}{\prod_{j=1}^{q}(c_j)_n (b')_n\, n\,!}\quad {}_3F_2\left[\begin{matrix}-n, & 1-b'-n, & 1-b'_1-n, \\ & b, & b'\end{matrix}\quad -\frac{u}{v}\right]$$

Which yield a number of interesting generating formulas for Jacobi and related Polynomial.

A large number of special cases have been discussed.

1. **Introduction :** The aim of this paper is to obtain some generating functions for Gegenbauer and generalised Rice Polynomial in terms of Kampe de Feriet double hypergeometric function of superior order. On employing on arbitrary sequence it yields several interesting generating relations.

The results proved here lead to certain generalization of various well known formulas due to Dhawn (5), Brafman (3), Carlitz (4) and several other results also.

The Kampe de Feriet's double hypergeometric function of superior order (1, P. 150) is defined by ;

$$F\left[\begin{array}{c|c|c} p & a_1,\ldots,a_p & \\ q & b_1,b'_1\ ..;\ b_a,b'_q & x, \\ r & c_1,\ ..;\ c_r & y \\ s & d,\ d'_1;\ldots;\ d_s,\ d'_s & \end{array}\right]=\sum_{m,n=0}^{\infty}\frac{\prod\limits_{j=1}^{p}(a_j)_{m+n}\ \prod\limits_{j=1}^{q}\{(b_j)_m(b'_j)_n\}}{\prod\limits_{j=1}^{r}(c_j)_{m+n}\ \prod\limits_{j=1}^{s}\{(d_j)_m(d'_j)_n\}}\times\frac{x^m y^n}{m!\,n!} \quad \ldots \quad (1.1)$$

$$|x|<1,\ |y|<1$$

We require the following results in the investigation.

$$(n-k)!=\frac{(-1)^k\,n!}{(-n)_k}\ ,0\leqslant k\leqslant n\ ; \quad \ldots \quad (1.2)$$

$$(a)_{n-k}=\frac{(-1)^k(a)_n}{(1-a-n)_k}\ ,\ 0\leqslant k\leqslant n\ ; \quad \ldots \quad (1.3)$$

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}A(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n-k) \quad \ldots \quad (1.4)$$

The Jacobi Polynomial $P_n^{(\alpha,\beta)}(x)$ is defined by (9, p. 254)

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n\,!}\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{array}{l}-n,\ -\beta-n\ ;\ \dfrac{x-1}{x+1}\\ \qquad 1+\alpha\ ;\end{array}\right] \quad \ldots \quad (1.5)$$

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\beta)^n}{n\,!}\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{array}{l}-n,\ -\alpha-n\ ;\ \dfrac{x+1}{x-1}\\ \qquad 1+\alpha\ ;\end{array}\right] \quad \ldots \quad (1.6)$$

The Gegenbaur polynomial is defined by (9, page 279)

$$C_n^{\nu}(x)=\frac{(2\nu)_n}{n\,!}\left(\frac{x+1}{2}\right)^n {}_2F_1\left(\begin{array}{l}-n,\ \frac{1}{2}-\nu-n\ ;\ \dfrac{x-1}{x+1}\\ \qquad \nu+\frac{1}{2}\ ;\end{array}\right) \quad \ldots \quad (1.7)$$

and the Legendre polynomial for $U=\frac{1}{2}$ is

$$P_n(x)=\left(\frac{x-1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n\,;\ -n\,; \\ 1\,;\end{matrix}\ \frac{x+1}{x-1}\right] \qquad \ldots \quad (1.8)$$

On reversing the order of Summation of (1.8)

$$P_n(x)=\left(\frac{x+1}{2}\right)^n {}_2F_1\left[\begin{matrix}-n\,;\ -n\,; \\ 1\,;\end{matrix}\ \frac{x-1}{x+1}\right] \qquad \ldots \quad (1.9)$$

In 1958 Bendit (2) introduced the following Polynomials

$$R_n(a,b,t)=\frac{(a)_n\,u^n}{n\,!}\ {}_3F_2\left[\begin{matrix}-n,\ 1-b-n,\ a\,; \\ b,\ 1-a-n\,;\end{matrix}\ -\frac{v}{u}\right]$$

$$G_n(a,b,t)=\frac{(a)_n\,u^n}{n\,!\,(a+b)_n}\ {}_3F_2\left[\begin{matrix}-n,\ a,\ b\,; \\ 1-a-n,\ 1-b-n\,;\end{matrix}\ -\frac{v}{u}\right]$$

Where $u=t-(t^2-1)^{\frac{1}{2}}$ and $v=t+(t^2-1)^{\frac{1}{2}}$.

The Tchebicheff Polynomials are defined by (9, p. 301)

$$T_n(x)=\frac{n\,!}{(\frac{1}{2})_n}\ P_n^{(-\frac{1}{2},\ -\frac{1}{2})}(x),\quad u_n(x)=\frac{(n+1)!}{(\frac{3}{2})_n}\ P_n^{(\frac{1}{2},\ \frac{1}{2})}(x).$$

2. **Let us Consider**

$$\sum_{m,n=0}^{\infty}\frac{C_{m+n}\,(ut)^m\,(vt)^n}{(b)_m\,(b')_n\,m!\,n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\frac{(C)_n\,u^m t^n v^{n-m}}{(b)_m\,(b')_{n-m}\,m!\,(n-m)!}$$

$$=\sum_{n=0}^{\infty}\frac{(C)_n\,(vt)^n}{(b')_n\,n!}\ {}_2F_1\left[\begin{matrix}-n,\ 1-b'-n\,;\ b\,;\end{matrix}\ \frac{u}{v}\right] \qquad \ldots \quad (2.1)$$

If we replace, C_{m+n} by $\left[\dfrac{\prod_{j=1}^{p}(a_j)_{m+n}}{\prod_{j=1}^{q}(b_j)_{n+m}}\right]$.

$$F\left[\begin{array}{c|c|c} p & a_1,\dots,a_p & \\ 0 & \dots & ut, \\ q & c_1,\dots,c_q & vt \\ 1 & b, b' & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n (vt)^n}{\prod_{j=1}^{q} (c_j)_n (b')_n} \; {}_2F_1\left[\begin{matrix} -n, 1-b'--n\,; \\ b\,; \end{matrix} \;\; \frac{u}{v}\right] \dots$$

If, $b=1+\alpha$, $b'=(1+\beta)$, $u=\dfrac{x-1}{2}$ and $v=\dfrac{1+x}{2}$

$$F\left(\begin{array}{c|c|c} p & a_1, \dots, a_p & \left(\frac{x-1}{2}\right) t, \\ 0 & \dots & \\ q & c_1, \dots, c_q & \left(\frac{x+1}{2}\right) t \\ 1 & 1+\alpha, 1+\beta & \end{array}\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n \, t^n}{\prod_{j=1}^{q} (c_j)_n \, (1+\beta)_n \, n!} \times \quad \dots \quad (2.2)$$

$$\left(\frac{(1+x)\,t}{2}\right)^n \; {}_2F_1\left[-n, -\beta-n\,;\, 1+\alpha\,;\, \frac{x-1}{x+1}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n}{\prod_{j=1}^{q} (c_j)_n} \; \frac{t^n}{(1+\beta)_n \, (1+\alpha)_n} \, . \, P_n^{(\alpha,\beta)}(x).$$

$$\left|\frac{x-1}{2} \, . \, t\right| < 1 \, , \; \left|\frac{x+1}{2} \, . \, t\right| < 1.$$

3 consider $\displaystyle\sum_{n,m=0}^{\infty} \frac{C_{m+n} \, (ut)^m \, (vt)^n}{(b)_m \, (b')_n \, m! \, n! \, (b_1)_m \, (b'_1)_n}$

$$= \sum_{n=0}^{\infty} \sum_{n=0}^{m} \frac{(C)_n \, V^n \left(\frac{u}{v}\right)^m t^n}{(b)_m \, (b')_{n-m} \, (n-m)! \, m! \, (b_1)_n \, (b'_1)_{n-m}} \, .$$

$$= \sum_{n=0}^{\infty} \frac{(C)_n \, (vt)^n}{n! \, (b')_n \, (b'_1)_n} \; {}_3F_2\left[\begin{matrix} -n\,; & 1-b'-n, & 1-b'_1-n\,; \\ & b_1 & b'_1\,; \end{matrix} \;\; -\frac{u}{v}\right]$$

If we replace ;

$$C_{m+n} \text{ by } \frac{\prod_{j=1}^{p} (a_j)_{n+m}}{\prod_{j=1}^{q} (C_j)_{n+m}} .$$

$$F\left(\begin{array}{c|c|c} p & a_1, \dots, a_p & \\ 0 & \dots & ut \\ q & c_1, \dots ; c_q & vt \\ 2 & b,b' ; b_1,b', & \end{array}\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n \, (vt)^n}{n! \prod_{j=1}^{q} (C_j)_n} < \frac{1}{(b'_1)_n \, (b')_n} \quad \dots \quad (3.1)$$

$${}_3F_2\left[\begin{matrix} -n, 1-b'-n, 1-b_1'-n \\ b, \quad b_1 ; \end{matrix} - \frac{u}{v}\right]$$

$$| ut | < 1, \; | vt | < 1,$$

Hence,

$$F\left(\begin{array}{c|c|c} p & a_1 ; \dots ; a_p & \\ 0 & \dots & ut, \\ q & c_1 ; \dots ; c_q & vt \\ 2 & bb' ; \;\; b_1 b_1' & \end{array}\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n \, (vt)^n}{\prod_{j=1}^{q} (C_j)_n \, (b'_1)_n \, (b'_n)_n \, (b)_n} \times \quad \dots \quad (3.2)$$

$$H_n^{(b-1,\, 1-b'-b-2n)}\left[1-b'-n, b_1, - \frac{u}{v}\right]$$

$$| ut | < 1, \; | vt | < 1.$$

If we replace $u=-x, v=1$.

$$\left(\begin{array}{c|c|c} p & a_1, \dots, a_p & \\ 0 & \dots & -xt \\ q & c_1, \dots, c_q & t \\ z & b,b ; b_1,b', & \end{array}\right) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n \, (vt)^n}{\prod_{j=1}^{q} (c_j)_n \, (b)_n \, (b')_n \, (b)_n} \times \quad \dots \quad (3.3)$$

$$H_n^{(b-1,\, 1-b'-b-2n)} [1-b'-n, b ; x]$$

where

$$H_n^{(\alpha,\beta)} [\epsilon, p ; v] = \frac{(1+\alpha)_n}{n!} \; {}_3F_2 \left[\begin{matrix} -n, 1+\alpha+\beta+n . \epsilon ; \\ 1+\alpha, p ; \end{matrix} v \right]$$

is generalize Rice's polynomial (8)

If we put, $b'=b=a \qquad b_1=b_1'=b$

and replace u by v and v by u, we get,

$$F\left[\begin{array}{c|c|c} p & a_1, \dots, a_p & \\ 0 & \dots & -vt \\ q & c_1, \dots, c_q & ut \\ 2 & a,a,\ \ b_1,b_1 & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n\,(ut)^n}{\prod_{j=1}^{q} (c_j)_n\, n!} \times \qquad \dots \quad (3.4)$$

$$\frac{1}{(b)_n(a)_n}\ {}_3F_2\left[\begin{matrix} -n,\ 1-a-n,\ 1-b-n\ ; \\ a,\ b \end{matrix} - \frac{v}{u}\right]$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n\, t^n\, (-1)^n}{\prod_{j=1}^{q} (c_j)_n\, (b)_n\, (a)_n\, (a)_n} \quad R_n(1-a-n,b,t).$$

where $u=t-(t^2-1)^{\frac{1}{2}}$ and $v=t+(t^2-1)^{\frac{1}{2}}$

$$F\left[\begin{array}{c|c|c} p & a_1, \dots, a_p & \\ 0 & \dots & -vt \\ q & c_1, \dots, c_q & ut \\ 2 & a,a,\ \ b,b & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{q} (a_j)_n\, t^n\, (-1)^n\, (2-a-b-n)_n}{\prod_{j=1}^{q} (b_j)_n\, (a)_n\, (b)_n\, (a)_n} \times \qquad \dots \quad (3.5)$$

$$G_n(1-a-n,\ 1-b-n,\ t).$$

when $u=t-(t^2-1)^{\frac{1}{2}}$ and $v=t+(t^2-1)^{\frac{1}{2}}$

$|ut| < 1$ and $|vt| < 1$.

4. Particular Cases

In (2.2) replacing $u=\dfrac{1-x}{2}$ and putting $b=b'=d+\frac{1}{2}$ and applying (1.7) we get—

$$F\left[\begin{array}{c|c|c} p & a_1, \dots, a_p & \dfrac{(1-x)}{2}\,.\,t, \\ 0 & \dots & \\ q & c_1, \dots, c_q & \dfrac{(1+x)}{2}\ t \\ 1 & d+\frac{1}{2},\ d+\frac{1}{2} & \end{array}\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (a_j)_n\, t^n}{\prod_{j=1}^{q} (c_j)_n\, (d+\frac{1}{2})_n\, (2d)_n} \quad C_n^d(x), \quad .. \quad (4.1)$$

$$\left|\frac{1-x}{2}\right| < 1 \text{ and } \left|\frac{1+x}{2}\,t\right| < 1.$$

$$b'=b=1,\ u=\frac{1-x}{2},\ v=\frac{1+x}{2}$$

$$F\left[\begin{array}{c|c|c} p & a_1,\ldots,a_p & \frac{(1-x)}{2}t, \\ 0 & \ldots & \\ q & c_1,\ldots,c_q & \frac{(1+x)}{2}t \\ 1 & 1,1 & \end{array}\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}P_n(x)\,t^n \qquad \ldots \quad (4.2)$$

$$\left|\left(\frac{1-x}{2}\right)t\right|<1,\ \left|\left(\frac{1+x}{2}\right)t\right|<1.$$

Putting $\beta=\alpha=\frac{1}{2}$ in [2.3] we get,

$$F\left[\begin{array}{c|c|c} p & a_1,\ldots,a_p & \left(\frac{x-1}{2}\right).\,t, \\ 0 & \ldots & \\ q & c_1,\ldots,c_q & \left(\frac{x+1}{2}\right).\,t \\ 1 & 3/2,\quad 3/2 & \end{array}\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,t^n\,u_n(x)}{\prod_{j=1}^{q}(c_j)_n\,(3/2)_n\,(n+1)!} \qquad \ldots \quad (4.3)$$

$$\left|\frac{x-1}{2}\right|<1,\ \left|\frac{x+1}{2}\,t\right|<1.$$

putting $\beta=\alpha=-\frac{1}{2}$ in [2.2], we get,

$$\left[\begin{array}{c|c||c} p & a_1,\ldots,a_p & \frac{x+1}{2}.\,t \\ 0 & \ldots & \\ q & c_1,\ldots,c_p & \frac{x-1}{2}.\,t \\ 2 & \frac{1}{2},\frac{1}{2} & \end{array}\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_n\,t^n}{\prod_{j=1}^{q}(c_j)_n\,(\frac{1}{2})_n\,n!}T_n(x) \qquad \ldots \quad (4.4)$$

$$\left|\frac{x-1}{2}\,t\right|<1;\ \left|\frac{x+1}{2}.\,t\right|<1.$$

In (4·1) setting $p=2$, $q=0$, the result (4·1) degenerates in the known result

Choosing $p=1$, $q=0$, we get—

$$\phi\left(a,\ d+\frac{1}{2},\ d+\frac{1}{2},\ \frac{x+1}{2}.\,t,\ \frac{x-1}{2}.\,t\right)=\sum_{n=0}^{\infty}\frac{(a)_n}{(d+\frac{1}{2})_n\,(2d)_n}C_n^d(x)\,t^n \qquad \ldots \quad (4.5)$$

Setting $a=d+\frac{1}{2}$ in (4·5) reduces to known formula

Putting $p=0$, $q=1$ in (2·1)

$${}_0F_1\left(-\,;\,d+\frac{1}{2}\,;\,\frac{x+1}{2}.\,t\right)\,{}_0F_1\left(-\,;\,d+\frac{1}{2}\,;\,\frac{x+1}{2}\,.\,t\right)=\sum_{n=0}^{\infty}\frac{C_n^d\,(x).t^n}{(d+\frac{1}{2})_n\,(2d)_n}$$

In (2·1) setting $p=2$, $q=0$, $a=b=d+\frac{1}{2}=\alpha+1$ we get—

$$F_4\left[\alpha+1\,;\,\alpha+1\,;\,\alpha+1\;\alpha+1\,;\,\frac{x-1}{2}\,.\,t\,,\,\frac{x+1}{2}\,.\,t\right]=\sum_{n=0}^{\infty}P_n^{(\alpha,\beta)}\,(x)\,.\,t^n$$

$$=\frac{2^n}{\omega(1-2x\,t+\omega)}$$

where ω stands for $(1-2xt+t^2)^{\frac{1}{2}}$ putting $p=2$, $a=0$, $a=b=1$, we get,

$$F_4\left[a,\,b\,;\;1,\,1\,;\,\frac{x+1}{2}\,.\,t\,,\,\frac{x-1}{2}\,.\,t\right]=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{n!\,n!}P_n\,(x)\,t^n$$

which is known formula (5, P 11 Eqn. 3·2)

In (4·1) taking $p=2$, $q=0$

$$F_4\left[a,\,b,\;d+\frac{1}{2}\,,\,d+\frac{1}{2}\,,\,\frac{x+1}{2}\,.\,t\,,\,\frac{x-1}{2}\,.\,t\right]=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(d+\frac{1}{2})_n\,(2d)_n}C_n^d\,(x)\,t^n$$

which is known formula (5 P 11 Eqn. 2.1)

In (4·3) taking $p=2$, $q=0$

Acknowledgements : The author expresses his gratitude to Dr. Rattansingh Prof. of Mathematics, Govt. College of Science, Raipur (M.P.) for his valuable guidance and help in the execution of this paper.

The author is thankfull to Dr. P. R. Khandekar Principal, Govt. College of Science, Raipur for providing all facilities in Connection with this work.

References

Appell, P. and Kampe de Feriet (1926) : Functions hypergeometric quies et hypersphriques Gauthier Villars Paris.

Bedient, P. E. (1958) ; Polynomial related to Appell function of two variable michigan thesis.

Brafman, F. (1936) : Generating functions of Jacobi and related Poly. Proc. *Amer. Math. Soc.*, 2(6), 948.

Carlitz ——— : A bilinear generating function for the Jacobi-Polynomial Poll Un Mat Flat (3) 18 (=1863) 87-89 MR 27 3841.

Dhavan, G. K. and Paliwal, D. D. (1976) : *The Mathematics Education* No. 1 X.

Erdelyie, A. (1953) : Higher Transcendental Function Vol. I

Jain, R. N. —— : Labnew Journal of Science and Technology, Kanpur, India, **8** **2, April 1970 P. 97.**

Khandekar, P. R. (1964) : On a generalization of Rice Polynomial-1 *Proc. Mat. Acod. Sci. India Sect*. A **34,** 157.

Rainville, E. D. (1960) Special function New York.

DEPARTMENT OF MATHEMATICS
GOVT. COLLEGE OF SCIENCE
RAIPUR, M.P., INDIA.

Bull Cal. Math. Soc., **75**, 363-368 (1983)

A NOTE ON D-SPACES

U. V. FATTEH and DARSHAN SINGH

(*Received 3 August 1981 ; Revision Received 12 July 1982*)

1. Introduction : Levine (1968) introduced the notion of a D-space in which every non-void open set is dense. He also proved a few properties of D-spaces in that paper.

In this paper we have established further characterizations and properties of D-spaces. We have also introduced and studied locally D-space.

In this paper $\bar{A}$, A^i, $X-A=A'$ and A^b denote closure interior, complement and boundary respectively of a subset A of a topological space X.

2. Preliminaries : Levine (1963) defined a semi-open subset in a top. space as a set A such that there exist an open set 0 so that $0 \subset A \subset \bar{0}$ and established the following results :

THEOREM 2.1 : *Open sets are semi open.*

THEOREM 2.2 : *A is semi open iff* $A \subset \bar{A}^i$.

Crossley and Hildebrand (1971) called the complement of a semi-open set to be a semi-closed set and proved that

THEOREM 2.3 : *A is semi-closed iff* $\bar{A}^i \subset A$

3. Characterizations of a D-space.

THEOREM 3.1 (*Levine 1968*) : *A space is a D-space iff it does not have two non-void disjoint open sets.*

THEOREM 3.2 : *Let X be a topological space. The following statements are equivalent.*

(i) X is a D-space.

(ii) X has no proper regularly open set or equivalently X has no proper regularly closed set.

(iii) X has no proper disjoint open sets G and H such that $X = \bar{G} \cup H = G \cup \bar{H}$

(iv) X has no proper closed sets F and K such that $X = F \cup K$; $F^i \cap K = F \cap K^i = \phi$.

(v) Every closed set F in X such that $F \neq X$ is nowhere dense in X.

(vi) X has no proper open set which is also semi-closed or equivalently X has no proper closed set which is semi-open.

Proof : (i) $\Rightarrow$ (ii) :

If X has a proper regularly open set 0 then $\overline{0} \neq X$. So X is not a D-space which is a contradiction. We can similarly prove that X can not have a proper regularly closed set.

(ii) $\Rightarrow$ (iii) :

If there exist proper disjoint open subsets G and H of X satisfying $X = \overline{G} \cup H = G \cup \overline{H}$, then $\overline{G}$ is a non-empty regularly closed set in X.

Also $\overline{G} \cap H = \phi$ which implies $\overline{G} \neq X$. Hence X has a proper closed set.

(iii) $\Rightarrow$ (iv) :

If X has proper closed subsets F and K such that $X = F \cup K$, $F^{i} \cap K = F \cap K^{i} = \phi$ then $G = F'$ and $H = K'$ are disjoint non empty open sets such that $X = \overline{G} \cup H$.

(iv) $\Rightarrow$ (i). Suppose there is a non empty open set 0 such that $\overline{0} \neq X$. Then $\overline{0}^{i} \neq X$. Take $K = \overline{0}^{i}$ and $F = (\overline{0}^{i})$.

Then X has F and K as in (iv).

(i) $\Rightarrow$ (v) :

It is obvious.

(ii) $\Rightarrow$ (vi) :

If 0 is a proper open sub set of X which is semi-closed then 0 is regularly open.

(vi) $\Rightarrow$ (ii) :

A regularly open set is open and semi-closed.

(See theorem 2.3).

Corollary 3.1 : *If $(X, \mathcal{T})$ is a top. space and $\mathcal{T}$ is finite then $(X, \mathcal{T})$ is a D-space iff intersection of all non-void open subsets of X is non void.*

Remark 3.1 : Above result is not true in case the topology on X is infinite.

For example a countably infinite space X with finite complements topology is a D-space but intersection of all non-empty open subsets of X is void.

Corollary 3.2 : *Every continuous function from a D-space into a T_2-space is a constant.*

Proof : This follows easily from Theorem 3.1.

DEFINITION 3.1 : *If X is a topological space then a pair of non-empty closed subsets F and K of X is said to form a D- disconnection of X if $X = F \cup K$, $F^i \cap K = F \cap K^i = \phi$.*

COROLLARY 3.3 : *If closed subsets F and K form a D-disconnection of X such that $F^b \cap K^b = \phi$ then X is disconnected.*

Hint : As F and K are closed, $F^i \cap K = F \cap K^i = \phi$ implies $F \cap K = \phi$.

Levine (1968) has proved the following results :

THEOREM 3.3 : (i) *If $f : X \rightarrow Y$ is continuous and X is a D-space then $f(X)$ is a D-space.*

(ii) *A product of topological spaces is a D-space iff each factor is a D-space.*

4. D-subsets of a topological space.

DEFINITION 4.1 : *A subset of a topo. space X is said to be a D-subset of X if it is a D-space as a subspace of X.*

THEOREM 4.1 : *If $A \subset Y \subset X$ then A is a D-subset of the space X iff it is a D-subset of the subspace Y of X.*

Proof is immediate from Theorem 3.1 and definition 4.1.

THEOREM 4.2 : *If A is a D-subset of X and F and K form a D-disconnection of X then $A \subset F$ or $A \subset K$.*

Proof : If $A \cap F \neq \phi$ and $A \cap K \neq \phi$ then it would be a D-disconnection of A. Therefore $A \cap F = \phi$ or $A \cap K = \phi$ together with $X = F \cup K$. Hence $A \subset F$ or $A \subset K$.

THEOREM 4.3 : *If A is an open D-subset of X and 0 is any non-void regularly open set in X then $A \subset 0$ or $A \subset X\text{-}0$.*

Proof : If $0 = X$ then the result follows.

If $0 \neq X$ then $F = \bar{0}$ and $K = X\text{-}0$ form a D-disconnection of X. So by the above theorem, $A \subset F$ or $A \subset K$ which implies $A \subset F^i = \bar{0}^i = 0$ or $A \subset K^i = (X-0)^i \subset \overline{(X-0)^i} = X = 0$.

THEOREM 4.4 : *If $\{ 0_\alpha : \alpha \in \Delta \}$ is a collection of open D-subsets of X and $\bigcap_{\alpha \in \Delta} 0_\alpha \neq \phi$ then $\bigcup_{\alpha \in \Delta} 0_\alpha$ is a D-subset of X.*

Proof : If $Y = \bigcup_{\alpha \in \Delta} 0_\alpha$ is not a D-subset of X then there exist a proper reg. open set G in the subspace Y of X. Since each 0_α is an open, D- subset of X, each 0_α is so in Y also.

Therefore by the previous theorem $0_\alpha \subset G$ or $0_\alpha \subset X-G$ for every $\alpha \in \Lambda$.

Let $x_0 \in \bigcap_{\alpha \in \Delta} 0_\alpha$. Then $x_0 \in G$ implies $Y=G$ and $x_0 \in X-G \Rightarrow Y \subset X-G \Rightarrow G \subset X-G \Rightarrow G=\phi$ which are prohibitive.

Following results can be easily proved.

COROLLARY 4.1 : *If* $\{ 0_i \mid i=1,2,3\ldots \}$ *is a family of open D-subsets of X and* $0_i \cap 0_{i+1} \neq \phi$ *for each i then* $\bigcup_{i=1}^{\infty} 0_i$ *is a D-subset of X.*

COROLLARY 4.2 : *If* $\{ 0_\alpha \}_{\alpha \in \Lambda}$ *is a family of open, D-subsets of X and if for some* $\alpha_0 \in \Lambda$, $0_{\alpha_0} \cap 0_\alpha \neq \phi$ *for every* α, *then* $\bigcup_{\alpha \in \Delta} 0_\alpha$ *is a D-subset of X.*

THEOREM 4.5 : *If A and B are D-subsets of X and* $A^i \cap B \neq \phi$ *or* $A \cap B^i \neq \phi$ *then* $A \cup B$ *is a D-subset of X.*

Proof : Suppose $Y=A \cup B$ is not a D-subset of X, then there exist open sets G and H in X such that $Y \cap G \neq \phi$, $Y \cap H \neq \phi$ and $Y \cap G \cap H=\phi$.

As A and B are D-subspaces of X, the sets G and H both can not intersect A and both can not intersect B. Without loss of generality assume that $G \cap B=\phi$. Then $G \cap A \neq \phi$, $A \cap H=\phi$ and $H \cap B \neq \phi$.

Now if $A \cap B^i \neq \phi$ then $A \cap G$ and $A \cap B^i$ are non-empty disjoint open sets in the subspace A of X which is prohibitive as A is a D-subspace of X.

Similarly if $B \cap A^i \neq \phi$ then B is not a D-subset of X which is a contradiction.

THEOREM 4.6 : *If* $\{ A_\alpha \}_{\alpha \in \Delta}$ *is a collection of D-subsets of X and* $(\bigcap_{\alpha \in \Delta} A_\alpha)^i \neq \phi$ *then* $\bigcup_{\alpha \in \Delta} A_\alpha$ *is a D-subset of X.*

Proof : Suppose $Y= \bigcup_{\alpha \in \Lambda} A_\alpha$ is not a D-subspace of X. Then there exist open sets G and H in X such that $Y \cap G \neq \phi$, $Y \cap H \neq \phi$ and $Y \cap G \cap H=\phi$. Therefore there exist β and γ in Λ such that $A_\beta \cap G \neq \phi$ and $A_\gamma \cap H \neq \phi$.

Case (i) If $\beta=\gamma$ then A_β is not a D-subspace of X.

Case (ii) If $\beta \neq \gamma$ then

$$\phi \neq (\bigcap_{\alpha \in \Delta} A_\alpha)^i \subset \bigcap_{\alpha \in \Lambda} A_\alpha^i \text{ implies } A_\beta^i \cap A_\gamma^i \neq \phi.$$

So $A_\beta^i \cap A_\gamma \neq \phi$. Therefore by the theorem 4.5

$A_\beta \cup A_\gamma$ is a D-subset of X. On the other hand $G \cap (A_\beta \cup A_\gamma)$ and $H \cap (A_\beta \cup A_\gamma)$ are disjoint non-empty open sets in $A_\beta \cup A_\gamma$.

THEOREM 4.7 : *Let* $\{A_\alpha\}_{\alpha \in \Lambda}$ *be a family of D-subsets of* X. *If for some* $\alpha \in \Lambda$, $A_{\alpha_0} \cap A_\alpha^i \neq \phi$ *for every* $\alpha \neq \alpha_0$ *then* $\underset{\alpha \in \Lambda}{U} A_\alpha$ *is a D-subset of* X.

Proof : Easy.

THEOREM 4.8 : *Let* $\{A_n \mid n=1,2,\ldots\}$ *be a sequence of D-subsets of* X *such that* $A_n^i \cap A_{n+1} \neq \phi$ *(or* $A_n \cap A_{n+1}^i \neq \phi$*) for all* n *then* $\overset{\infty}{\underset{n=1}{U}} A_n$ *is a D-subset of* X.

Proof : Follows from Theorem 4.5.

THEOREM 4.9 : *Suppose* X *is a D-space and* C *is a D-subset of* X. *Suppose further that subspace* $X-C$ *of* X *has an open set* V. *Then* $C\ U\ V$ *is a D-subset of* X.

Proof : If $Y=C\ U\ V$ is not a D-subspace of X then there exist open sets G and H in X such that $0=Y \cap G \neq \phi$, $S=Y \cap H \neq \phi$ and $0 \cap S=Y \cap G \cap H=\phi$. As C is a D-subset of X, either $C \cap G=\phi$ or $C \cap H=\phi$. Suppose $C \cap G=\phi$. Then $0 \subset V$. Therefore 0 is open in $X-C$. Also since 0 is open in Y, 0 is open in $Y\ U\,(X-C)=X$. Since $0 \cap H=\phi$ and $H \neq \phi$, $\bar{0} \neq X$. Thus $\bar{0}$ is a proper reg. closed subset of X. Hence X ceases to be a D-space.

DEFINITION 4.2 : *A subset* W *of a space* X *is said to be weakly open if there exists an open set* 0 *such that* $\bar{W}=\bar{0}$.

Remark 4.1 : All semi open sets are weakly open. Converse is not true. Clearly X is a D-space iff every non-null weakly open subset of X is dense.

THEOREM 4.10 : X *is a D-space iff every non-empty weakly open subset of* X *is a D-subset of* X.

Proof is simple.

5. A Locally D-space and Maximal D-subsets of a Space.

DEFINITION 5.1 : *A top. space* X *is said to be locally D at* $x \in X$ *is for any given open set* 0 *containing* x, *there is an open D-subset* G *of* X *such that* $x \in G \subset 0$.

X is said to be a locally D-space if it is locally D at every point of X.

Remark 5.1 : X is a locally D-space if it is a D-space.

This follows from Theorem 4.10.

Converse of this result is not true.

Note : This result is at variance with the corresponding relationship between local connectedness and connectedness.

THEOREM 5.1 : *If $f : X \rightarrow Y$ is continuous, open and onto and X is a locally D-space then so in Y.*

This follows from Theorem 3.3 (i).

THEOREM 5.2 *A product of finitely many locally D-spaces is a locally D-space.*

This follows from Theorem 3.3 (ii),

Remark : Above result is not true in case of infinitely many spaces.

Norman Levine has proved the following theorem.

THEOREM 5.3 (Levine 1968) : *Every top. space is the union of its maximal D-subsets.*

We shall prove :

THEOREM 5.4 : *If A and B are maximal D-subsets of X then either $A=B$ or $A^i \cap B=A \cap B^i=\phi$.*

Proof is immediate from Theorem 4.5.

COROLLARY 5.1 : *If all the maximal D-subsets of X are open then they form a partition of X.*

Proof : Easy.

THEOREM 5.5 : *X is a locally D-space iff each maximal D-subset of an open subspace of X is open.*

Proof : Necessity follows from Theorem 4.5 and sufficiency follows from Theorem 5.3.

References

Crossley and Hildebrand (1971) : 'Semi closure'. *The Texas Journal of Science*. **XXII**, 2 and 3 April 30.

Levine, N. (1968) : 'Dense Topologies'. *Amer. Math. Monthly*. **75**, 847.

Levine, N. (1963) : 'Semi open sets and semi continuity in top. spaces'. *Amer. Math. Monthly*. **70**, 36.

DEPARTMENT OF MATHEMATICS
GUJARAT UNIVERSITY
AHMEDABAD 380 009.

Bull. Cal. Math. Soc., **75**, 369-371 (1983)

ON A THEOREM IN REAL ANALYSIS

S. GHOSH

(Received 1 September 1981 ; 2nd Revision Received 17 May 1982 ; 3rd Revision Received 23 June 1983)

Abstract : A generalised form of a known theorem in real analysis is proved by a method more simple and elegant than an existing method.

Introduction : Here by use of a wellknown martingale convergence theorem (Diestel, J (1975), P 272), we have generalised a known theorem (Hobson (1927), P 627), which establishes the redundency of the Hellinger integral in the sense that such an integral corresponds to an equivalent Lebesgue integral. Our proof is more simple and elegant than the one cited above.

2. Notation :

Let (Ω, d) be a metric space. Let σ denote the Borel-σ-algebra on (Ω, d).

Let μ be a bounded positive measure defined on (Ω, σ). Let H be a real Banach-space, zero-element of which is denoted by θ and norm on which is denoted by

$$\|\cdot\|. \quad \text{Let } P \in [1, \infty).$$

Let $L_p(\Omega, \sigma, \mu) = \{ f : \Omega \to H, f$ is strongly measurable (with respect to σ) and Bochner-integrable (with respect to μ) (Diestel, J and, J. J. uhl, Jr. (1977) and Hille, E and Philips R. S. (1965)) and further

$$\int \|f\|^p \, d\mu < \infty \} \qquad \dots \quad (2.1)$$

Suppose $\forall$ integer $i \geqslant 1$, $\{A_{i1}, A_{i2}, \dots\}$ is a partition of Ω into pairwise disjoint σ-measurable sets $\ni \forall_i \geqslant 1$, $\{A_{i+11}, A_{i+12}, \dots\}$ is a refinement of $\{A_{i1}, A_{i2}, \dots\}$ and

$$\lim_{i \to \infty} \Big[\sup_{j :\, 1 \leqslant j < \infty \text{ and } A_{ij} \text{ is nonempty}} (\text{diameter of } A_{ij}) \Big] = 0 \qquad \dots \quad (2.2)$$

Simmons (1963), P 58). $\forall f \in L_p(\Omega, \sigma, \mu)$ and $\forall i \in \{1, 2, \dots\}$, denote

$$S_i(f) = \sum_{j=1}^{\infty} (\mu(A_{ij}))^{1-p} \left\| \int_{A_{ij}} f \, d\mu \right\|^p \qquad \dots \quad (2.3)$$

where if $\mu(A_{ij})=0$, then $(\mu(A_{ij}))^{1-p}\left\|\int_{A_{ij}} f\,d\mu\right\|^p$ should be interpreted as 0.

3. *The Theorem*

THEOREM 3.1

$$\forall f \in L_p(\Omega, \sigma, \mu),\ \lim_{i\to\infty} S_i(f)= \int \|f\|^p\,d\mu$$

Proof: Ommitting a trivial case, we assume that $\mu(\Omega) > 0$. $\forall\ x \in \Omega$ and $r \in (o,\infty)$, denote $S_{xr}=[y \in \Omega : d(y,x) < r]$. Let $x, \in \Omega\ r, \in (o,\infty)$. Let $y \in S_{x_1 r_1}$. $\exists$ integers $i \geqslant 1, j \geqslant 1 \ni y \in A_{ij} \subset S_{x_1 r_1}$. Thus $S_{x_1 r_1}$ is expressible as the union of a subclass of $\{A_{ij}, i \geqslant 1, j \geqslant 1\}$. $\forall\ i \in \{1,2,3,4, \ldots\}$, let σ_i=smallest σ-algebra with respect to which, each set A_{ij} is measurable. Since σ=smallest σ-algebra with respect to which each set S_{xr} (where $x \in \Omega$ and $r \in (o,\infty)$ is measurable, σ=smallest σ-algebra containing $\bigcup_{i=1}^{\infty} \sigma_i$.

Let $f \in L_p(\Omega, \sigma, \mu)$.

Let $i \in \{1,2,3,4, \ldots\ldots\}$.

Let $f_i(x)= \sum_{j=1}^{\infty} I_{A_{ij}}(x)\left[(\mu(A_{ij}))^{-1}\int_{A_{ij}} f\,d\mu\right]$, $\forall\ x \in \Omega$, where $\forall\ A \in \sigma$, I_A denotes characteristic function of the set A and, if $\mu(A_{ij})=0$, $(\mu(A_{ij}))^{-1}\int_{A_{ij}} f\,d\mu$ should be interprated as θ. Using the fact that, $\forall\ i \geqslant 1,\ j \geqslant 1,\ A_{ij}$ is a set-theoretic atom of σ_i, one can show that, $\forall\ i \geqslant 1,\ f_i=E^{\sigma_i} f$ *a.s.* $[\mu]$, where $E^{\sigma_i} f$ is the conditional expectation of f given σ_i in the probability-space $\left(\Omega, \sigma, \frac{\mu}{\mu(\Omega)}\right)$ (Diestel, J (1975), P 271). One may directly prove as a special case of Jensen's inequality for conditional expectation, (Proof of theorem 4, P 123, Diestel and uhl (1977)) that, $\|f_i\|^p \leqslant E^{\sigma_i}\|f\|^p$ *a.s.* $[\mu]$, and hence, the sequence $\{\|f_i\|^p, i=1,2, \ldots\}$ is uniformly integrable. By a known martingale convergence theorem (Diestel, J (1975), P 272),

$$\limsup_{i\to\infty} \int \big|\|f_i\| - \|f\|\big|\,d\mu \leqslant \lim_{i\to\infty} \int \|f_i - f\|\,d\mu=0$$

Hence

$$\forall\ \delta \in (o,\infty),\ \lim_{i\to\infty} \mu\{\omega \in \Omega : \big|\|f_i(\omega)\| - \|f(\omega)\|\big| > \delta\} = 0.$$

By a wellknown theorem (Neveu (1965), P 57),

$$\lim_{i\to\infty} S_i(f) = \lim_{i\to\infty} \int \| f_i \|^p \, d\mu = \int \| f \|^p \, d\mu.$$

Acknowledgement : The author is thankful to the referee who drew his attention to the relevant literature on vector measures and integration and brought out the significance of the result (in the context of Hellinger integral).

References

Diestel, J. (1975) : "Geometry of Banach Spaces", *Springer Verlag.*

Diestel, J. and J. J. Uhl, Jr. (1977) : "Vector measures", *Amer. Math. Soc.*, (Mathematical Survey Series).

Hille, E. and Philips, R. S. (1965) : "Functional analysis and semigroups", second printing of revised edition, *Amer. Math. Soc.* Colloquium Publications, XXXI.

Hobson, E. W. (1927) : "The theory of functions of a real variable and the theory of Fourier's series", I, Dover publications.

Neveu, J. (1965) : "Mathematical Foundations of the Calculus of probability", Holden Day, London.

Simmons, G. F. (1963) : "Introduction to Topology and modern analysis", Mcgraw Hill, New York.

SATYENDRANATH BOSE INSTITUTE OF PHYSICAL SCIENCES
CALCUTTA UNIVERSITY
and
DEPARTMENT OF STATISTICS
PRESIDENCY COLLEGE, CALCUTTA.

Bull. Cal. Math. 75, 373-375 (1983)

ON STRONG PERRON SUMMABILITY OF THE (c, 1) MEAN OF A FOURIER SERIES

L. M. TRIPATHI and V. N. TRIPATHI

(*Received 18 August 1981 ; Revision Received 27, September 1982*)

In this paper, we have proved a theorem on strong Perron summability $[P, 1]$ of the $(C, 1)$ mean of a Fourier series which generalises the result of Rao (1973) on strong (L) summability of the $(C, 1)$ mean of a Fourier series.

§1. Let Σa_n be an infinite series with the sequence $\{ S_n \}$ of its partial sums. Let $\{ p_n \}$ be a sequence of non-negative numbers and let the series $p(x) = \sum_{n=1}^{\infty} p_n x^n$ converge for all x, $0 < x < 1$ and $p(x) \uparrow \infty$ as $x \uparrow 1$. If

$$\sum_{n=1}^{\infty} | S_n - S |^k p_n x^n = 0[p(x)] \qquad \ldots \quad (1.1)$$

as $x \uparrow 1$, then the series Σa_n is said to be strongly Perron summable with index k, or, simply summable $[P. k]$ (Jain and Ganguly (1979)).

By (1.1), the $(C, 1)$ mean σ_n of the series Σa_n is $[p, k]$ summable to S if

$$\sum_{n=1}^{\infty} | \sigma_n - S |^k p_n x^n = o[p(x)], \text{ as } x \uparrow 1. \qquad \ldots \quad (1.2)$$

Let f be a $2\pi-$ periodic and Lebesgue integrable function of t in the interval $(-\pi, \pi)$ and let

$$f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) \qquad \ldots \quad (1.3)$$

be a Fourier series of the function $f(t)$.

With a fixed point x, let us write

$$\phi(t) = f(x+t) + f(x-t) - 2 f(x). \qquad \ldots \quad (1.4)$$

§2. Here, in this present note, we prove the following :

THEOREM : *If*

$$\text{(i)} \quad \int_0^t |\phi(u)|\, du = 0\left[\frac{p(1-t)}{p'(1-t)}\right], \text{ as } t \downarrow 0 \qquad \ldots \quad (2.1)$$

$$\text{(ii)} \quad \int_t^\pi \frac{|\phi(u)|}{u}\, du = 0\,[\,p(1-t)\,], \text{ as } t \downarrow 0 \qquad \ldots \quad (2.2)$$

and

$$\text{(iii)} \quad \sum_{n=1}^{\infty} n\, p_n\, y^n = 0\,[\,p'(y)]\ \text{as } y \uparrow 1 \qquad \ldots \quad (2.3)$$

then

$$\sum_{n=1}^{\infty} \left| \sigma_n(x) - f(x) \right| p_n\, y^n = 0\,[\,p(y)\,], \text{ as } y \uparrow 1 \qquad \ldots \quad (2.4)$$

It is easy to see that our theorem includes the result of Rao (1973) *for the case* $p_n = \frac{1}{n}$.

§3. *Proof of the theorem :* As in (Zygmund 1959), we have

$$\sigma_n(x) - f(x) = \frac{1}{n\pi} \int_0^\pi \frac{\sin^2 nt/2}{2 \sin^2 t/2}\, \phi(t)\, dt \qquad \ldots \quad (3.1)$$

Hence

$$\sum_{n=1}^{\infty} \left| \sigma_n(x) - f(x) \right| p_n\, y^n$$

$$= \sum_{n=1}^{\infty} p_n\, y^n \left| \frac{1}{n\pi} \int_0^\pi \frac{\sin^2 nt/2}{2 \sin^2 t/2}\, \phi(t)\, dt \right|$$

$$\leqslant \sum_{n=1}^{\infty} p_n\, y^n \frac{1}{n\pi} \left| \int_0^{1-y} \frac{\sin^2 nt/2}{2 \sin^2 t/2}\, \phi(t)\, dt \right|$$

$$+ \sum_{n=1}^{\infty} p_n\, y^n\, \frac{1}{n\pi} \left| \int_{1-y}^{\pi} \frac{\sin^2 nt/2}{2 \sin^2 t/2}\, \phi(t)\, dt \right|$$

$$= I_1 + I_2 \text{, say} \qquad \ldots \quad (3.2)$$

Now,

$$I_1 \leqslant \left\{ \sum_{n=1}^{\infty} n\, p_n\, y^n \right\} \int_0^{1-y} |\,\phi(t)\,|\, dt$$

$$= O\,[\,p'(y)\,]\,0\,\left[\frac{p(y)}{p'(y)}\right]$$

$$= 0\,[\,p(y)\,] \text{ as } y \uparrow 1. \qquad \ldots \quad (3.3)$$

Again

$$I_2 \leqslant \int_{1-y}^{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin nt/2}{nt/2}\right)^2 |\,\phi(t)\,|\, dt$$

$$= O\left[\int_{1-y}^{\pi} \frac{|\,\phi(t)\,|}{t}\, dt\right], \text{ (by (4.17), p. 10, Zygmund 1959).}$$

$$= 0\,[\,p(y)\,], \text{ as } y \uparrow 1. \qquad \ldots \quad (3.4)$$

Now, the result (2.4) follows from (3.3) and (3.4). This completes the proof of our theorem.

References

Jain, R. K. and Ganguly, A. (1979): Strong summability by logarithmic method of derived Fourier series and its conjugate series. *Ind. J. Pure and Appl. Math.*, **10(2)**, 250.

Rao, A. S. (1973): On strong (L) summability of the (*C*, 1) mean of a Fourier series. *Bull. Cal. Math. Soc.*, **65** (1), 15.

Zygmund, A. (1959): Trigonometric series, I, Second edition, Cambridge at the university press.

DEPARTMENT OF MATHEMATICS
BANARAS HINDU UNIVERSITY
VARANASI 221 005

Bull. Cal. Math. Soc., **75**, 377-380 (1983)

ON UNIFORM TRIANGULAR MATRIX SUMMABILITY OF LEGENDRE SERIES

L. M. TRIPATHI and V. N. TRIPATHI

(*Received 17 August 1981 ; Revision Received 27 September 1982*)

Dwivedi (1970) has established a theorem on uniform harmonic summability of Legendre series. In this paper, we propose to establish a corresponding result for uniform triangular matrix summability of Legendre series which includes the harmonic summability as one of its particular cases.

§1. An infinite series $\Sigma\, u_n(x)$ with the sequence of partial sums $\{S_n(x)\}$ is summable by triangular matrix method to $S(x)$, or simply $\wedge$-summable to $S(x)$ if

$$\sigma_n(x)=\sum_{k=0}^{n} \lambda_{n,k}\, u_k(x)=\sum_{k=0}^{n} \Delta\lambda_{n,k}\, S_k(x) \qquad \ldots \quad (1.1)$$

$\to S(x)$ as $n \to \infty$, where

$$\Delta\, \lambda_{n,k}=\lambda_{n,k}-\lambda_{n,k+1}$$

and $\{\, \lambda_{n,k}\, \}$ is a triangular matrix of real or complex numbers. (Hardy, 1949). If

$$\lim_{n\to\infty}\, [\sigma_n(x)-S(x)]=o(1) \qquad \ldots \quad (1.2)$$

uniformly in a set E, then the series $\Sigma u_n(x)$ is summable $(\wedge)$ uniformly in E to the sum $S(x)$.

Let us write

$$\Psi(t)=\Psi_\theta(t)=f\,\{\cos\,(\theta-t)\}-f\,(\cos\,\theta) \qquad \ldots \quad (1.3)$$

where $f\,(x)$ is a Lebesgue-integrable function in the range $(-1,1)$ associated with which is the Legendre series given by

$$f\,(x)\sim\sum_{n=0}^{\infty} a_n\, P_n(x) \qquad \ldots \quad (1.4)$$

where

$$a_n=(n+\tfrac{1}{2})\int_{-1}^{1} f\,(x)\, P_n(x)\, dx \qquad \ldots \quad (1.5)$$

and $P_n(x)$ is the n^{th} Legendre polynomial.

§2. In this note, we propose to establish the following :

THEOREM. *If*

$$\int_t^\eta \frac{|\Psi(u)|}{u^2}\,du = o\left[\frac{1}{t\log\frac{1}{t}}\right]. \qquad \dots \quad (2.1)$$

where $o \leqslant t < \eta < \pi$ is fixed, uniformly in a set E defined in the interval $(-1,1)$, in which $f(x)$ is bounded, as $t \to +0$; then the series (1.4) is summable $(\wedge)$ uniformly in E to the sum $f(x)$.

§3. *Proof of the theorem.* It can be easily observed that on partial integration the condition (2.1) implies

$$\int_0^t |\Psi(y)|\,dy = o\left[\frac{t}{\log\frac{1}{t}}\right] \qquad \dots \quad (3.1)$$

as $t \to +0$, uniformly in E

Now, the n^{th} partial sum of the series (1.4) is given by

$$t_n(x) - f(x) = \frac{1}{\pi\sqrt{\sin\theta}}\int_0^\eta \frac{f\{\cos(\theta-t)\} - f(\cos\theta)}{\sin\frac{t}{2}}\sin(n+1)t\sqrt{\sin(\theta-t)}\,dt + o(1), \qquad \dots \quad (3.2)$$

where $o < \eta \leqslant \delta < 1$, $x = \cos\theta$, $y = \cos\phi$, $o < \theta < \pi$, $\theta - \varphi = t$, $o < \varphi < \pi$, etc. Since $f(x)$ is bounded in the set E, so $o(1)$ will tend to zero for any x uniformly in E.

If $\sigma_n(x)$ be the $(\wedge)$-mean of the sequence $\{S_n(x)\}$ of the partial sums of the series (1.4), then

$$\sigma_n(x) - f(x) = \sum_{k=0}^{n} \triangle\lambda_{n,k}\,[t_k(x) - f(x)]$$

$$= O\left[\int_0^\eta \left\{\sum_{k=0}^{n} \Delta\lambda_{n,k}\frac{\sin(k+1)t}{\sin t/2}\right\}\{f\{\cos(\theta-t)\} - f(\cos\theta)\}dt\right]$$

$$+\ o(1), \text{ uniformly in } E$$

$$=O\left[\int_0^\eta |\Psi(t)|\,|N_n(t)|\,dt\right]+o(1), \text{ uniformly in } E.$$

$$=O\left[\int_0^{1/n}+\int_{1/n}^\eta\right]+o(1), \text{ uniformly in } E.$$

$$=O(I_1)+O(I_2)+o(1), \text{ uniformly in } E, \text{ say.} \qquad \dots \quad (3.3)$$

Now, let us consider I_1. It is easy to see that uniformly in $o \leqslant t \leqslant 1/n$,

$$|N_n(t)| = O(n), \qquad \dots \quad (3.4)$$

by Abel's transformation and the regularity condition of $(\wedge)$-summability (Hardy, 1949). Therefore,

$$I_1=\int_0^{1/n} |\Psi(t)|\,|N_n(t)|\,dt$$

$$=O(n)\int_0^{1/n} |\Psi(t)|\,dt, \quad \text{by (3.4)}$$

$$=O(n)\,o\left[\frac{1}{n\log n}\right], \quad \text{by (3.1)}$$

$$=o(1), \text{ uniformly in } E. \qquad \dots \quad (3.5)$$

Next, let us consider I_2. Here it can be seen by using lemma 2 and lemma 3 of Kishore and Hotta (1971) that

$$|N_n(t)| =O\left[\triangle\,\lambda_{n,\,n-\tau}\cdot\frac{1}{t^2}\right] o \leqslant t \leqslant \pi\,; \qquad \dots \quad (3.6)$$

and

$$\triangle\,\lambda_{n,0}=O\,(1/n) \qquad \dots \quad (3.7)$$

where $\tau=[1/t]$, the integral part of $1/t$.

Now,

$$I_2=\int_{1/n}^\eta |\Psi(t)|\,.\,|N_n(t)|\,dt$$

$$=O\left[\int_{1/n}^{\eta} |\Psi(t)| \,.\, \Delta\lambda_{n,n-\tau} \,.\, \frac{1}{t^2}\, dt\right] \text{ by (3.6)}$$

$$=O\left[\int_{1/n}^{\eta} \frac{|\Psi(t)|}{t^2} .\, \Delta\lambda_{n,0}\, dt\right]$$

$$=O\,(1/n)\,.\left[\int_{1/n}^{\eta} \frac{|\Psi(t)|}{t^2}\, dt\right] \text{by (3.7)}$$

$$=O\,(1/n)\,.\,o\left[\frac{n}{\log n}\right], \text{ by (2.1)}$$

$$=o(1), \text{ uniformly in } E. \qquad \dots \quad (3.8)$$

Thus, the result follows from (3.3), (3.5) and (3.8). This completes the proof of our theorem.

References

Dwivedi, G. K. (1970) : On the uniform harmonic and Nörlund summability of Legendre series. Chapter IX, Ph.D. thesis, B.H.U.

Hardy, G. H. (1949) : Divergent series, Oxford.

Kishore, N. and Hotta, G. C. (1971) : On absolute matrix summability of a Fourier series. *Indian Journal of Mathematics*, **13**, 2, 99.

DEPARTMENT OF MATHEMATICS
BANARAS HINDU UNIVERSITY
VARANASI 221 005.

Bull. Cal. Math. 75, 381-388 (1983)

CRITERIA DETERMINING THE STABILITY OF THE QUASI-EQUILIBRIUM SPECTRUM ENVISAZED THROUGH LINEARIZATION OF THE HYDRODYNAMICAL EQUATION OF MOTION IN ABSENCE OF PRESSURE-GRADIENT TERM

BHULU RANJAN SAHA and KSHETRA MOHAN GHOSH

(*Received 22 September 1981 ; Revision Received 18 November 1982*)

Abstract : In this paper, the similarity spectrum numerically calculated in a previous work by one of the authors (Saha, Eighth Casam-Cu-National Research Symposium, March, 1981) has been considered for the stability or otherwise of the entire phenomena. A small perturbation has been introduced in a small band, and the rate of disappearance of the said perturbation has been estimated and compared to the overall rate of energy dissipation pertaining to the original spectrum field. The ratio of the square of the life-time of the perturbation to that of the turbulence so envisazed is symbolized by γ^2. Now γ, calculated in the above manner, turns out to be less than unity (cf. Fig. 1) which reveals that the turbulence under consideration is unstable.

1. (a) *Symbols used.*

γ^2 = Ratio of the square of life time of perturbation to that of the turbulence.

$G(k,t)$ = Time spectrum,

$\phi(k,t)$ = Perturbation.

$G_e(k,t)$ = Quasi-equilibrium Spectrum.

$P(x)$ = Fraction of energy dissipation.

1. (b) Introduction :

According to definition, γ^2 is given by

$$\gamma^2 = \frac{\dfrac{1}{e}\,\dfrac{d^2 e}{dt^2}}{\dfrac{1}{\overline{u^2}}\cdot\dfrac{d^2\overline{u^2}}{dt^2}} \qquad \ldots \quad (0)$$

from which γ is calculated. Below $\gamma=1$, there is a considerable portion of energy but only a negligible fraction of the dissipation. For $\gamma < 1$, the spectrum is not expected to reach the quasi-equilibrium state i.e. in other words for $\gamma < 1$, the turbulence under consideration is said to be unstable.

For the equation relating to the decay of turbulence viz.

$$\frac{\partial^2}{\partial t^2}\int_0^k k^2 G(k,t)\,dk = \frac{4}{3}\int_k^\infty k^2 G(k,t)\,dk = \int_0^k k^4 G(k,t)\,dk, \qquad \dots (1)$$

[cf. equation (68), Saha 1981]

we have a self-preserving solution of the form

$$G(k,t) = \frac{3}{4k_0^5 t_0^2}\left(\frac{t_0}{t}\right)^{2-5c} g\left[\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c\right] \qquad \dots (2)$$

[cf. equation (72), Saha 1981]

which obeys

$$\int_0^x (3-2c)(2-2c)x^2 g\,dx - c(5-7c)x^3 g + c^2 x^4 g'$$

$$= \int_x^\infty x^2 g\,dx \int_0^\infty x^4 g\,dx \qquad \dots (3)$$

[cf. equation (71), Saha 1981]

For $c = \frac{2}{7}$, the equation (3) is reducible to the form

$$2h^2 h''' + 6hh'h'' - 39hh'' - 4h'^2 + 70h' - 294 = 49y - 49h \qquad \dots (4)$$

[cf. equation (9), Saha 1981].

2. Stability Criteria of the above Solutions :

Considering the equation (1) and following C. C. Lin (1953), we superimpose a perturbation $\phi(k,t)$ on the quasi-equilibrium spectrum $G_\varepsilon(k,t)$ only for the width of the band (k_*, k_*+dk_*). The function $\phi(k,t)$ will obviously obey the same equation as

$$\frac{\partial^2}{\partial t^2}\int_0^k k^2\phi(k,t)\,dk = C\int_k^\infty k'^2\phi(k't)\,dk' \int_0^k k'^4 G_e(k',t)\,dk'$$

$$+ \frac{4}{3}\int_k^\infty k'^2 G_\varepsilon(k',t)\,dk' \int_0^k k^4\phi(k,t)\,dk \qquad \dots (5)$$

Let us delimit ourselves to the estimation of the initial rate of disappearance of the perturbation when it is superposed on a narrow band in the neighbourhood of the wave-number k_* (say). If e denotes the total energy of the perturbation, we easily obtain from (5),

$$\frac{1}{e} \cdot \frac{d^2 e}{dt^2} = \frac{4}{3} k_*^2 \int_{k_*}^{\infty} k^2 G_c(k,t)\, dk \qquad \dots (6)$$

where

$$e = \int_{k_*}^{k_* + dk_*} \phi(k,t)\, dk.$$

[cf. Lin 1953]

This is to be compared to (8) derived from

$$\frac{1}{\overline{u^2}} \cdot \frac{d\, \overline{u^2}}{dt} = -\frac{10\nu}{\lambda^2}, \qquad \dots (7)$$

where

$$\lambda^2 = 7\nu t \quad \text{for} \quad c = \tfrac{2}{7}.$$

From (7) with $\lambda^2 = 7\nu t$, we can derive the equation

$$\frac{1}{\overline{u^2}} \cdot \frac{d^2 \overline{u^2}}{dt^2} = \frac{170}{49 t^2}. \qquad \dots (8)$$

We can now introduce γ^2 as

$$\gamma^2 = \frac{\dfrac{1}{e} \cdot \dfrac{d^2 e}{dt^2}}{\dfrac{1}{\overline{u^2}} \cdot \dfrac{d^2 \overline{u^2}}{dt^2}}.$$

Hence, from (6) and (8), we have

$$\gamma^2 = \frac{98}{255} k_*^2 t^2 \int_{k_*}^{\infty} k^2 G_c(k,t)\, dk, \qquad \dots (9)$$

which is obtained with the help of substitution of the form

$$G_c(k,t) = \frac{3}{4 k_0^5 t_0^2} \left(\frac{t_0}{t}\right)^{2-5c} g\left[\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c\right].$$

Using the relations

$$x=\frac{k}{k_0}\left(\frac{t}{t_0}\right)^c \quad \text{and} \quad c=\frac{2}{7},$$

we finally obtain

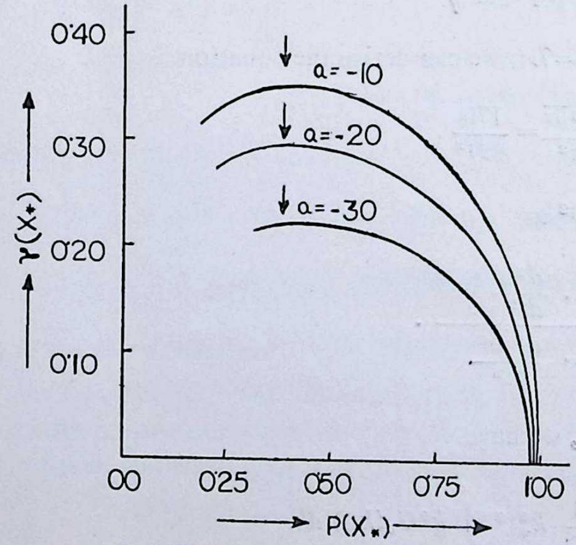
$$\gamma^2(x_*)=\frac{49}{170}\,x_*^2\int_{x_*}^{\infty} x^2 g(x)\,dx$$

i.e.

$$\gamma(x_*)=0.5369\;x_*\sqrt{\int_{x_*}^{\infty} x^2 g(x)\,dx}\,. \qquad \ldots \quad (10)$$

Again the fraction of energy dissipation below x_* is given by

$$P(x_*)=\frac{\int_0^{x_*} x^2 g(x)\,dx}{\int_0^{\infty} x^2 g(x)\,dx}\,. \qquad \ldots \quad (11)$$



THREE P(X*), γ(X*) CURVES FOR
THREE VALUES OF THE PARAMETER
a_2=-10, -20, & -50
Fig :1

Now we can use the numerical tables for $x^2g(x)$ with parametric values $a_2 = -10, -20, -50$ to construct the sets of tables for $P(x_*)$ and $\gamma(x_*)$.

From the tables of values so obtained, we can have graphical plotting of $P - \gamma$ curves (cf. Fig. I). As the graphs so drawn are below the level $\gamma = 1$, the turbulence thus envisazed in $[x, x^2g(x)]$ tables are unstable.

3. **Conclusions :** In the initiation stage of turbulence, the viscous terms are insignificant and there develop unstable stages of turbulence pictured through $c = \frac{2}{7}$, which leads to asymptotic behaviour $F(k,t) \sim k^4$ for $k \to 0$. Such instability continues till the turbulence reaches a stable stage when viscous terms start contributing at $c \doteq \frac{1}{2}$. The latter case is visualized in the works of S. Chandrasekhar (1949) which needs truncation of the similarity spectrum at the left end consequently demanding a curve fitting at that left part end of the spectrum.

Table 1—$[P(x_*), \gamma(x_*)]$ table corresponding to $(x, f(x))$ tables for parametric value $a_2 = -10$

x	$x^2g(x) = f(x)$	$P(x_*)$	$\gamma(x_*)$
.673197	1.318563	.1757	.3056
.727949	1.683418	.2617	.3398
.768473	1.933288	.3303	.3499
.794198	2.065348	.3916	.3521
.817526	2.158031	.4345	.3508
.846281	2.228249	.4757	.3475
.866754	2.245275	.5283	.3414
.886540	2.235934	.5666	.3350
.899808	2.213798	.6036	.3279
.919402	2.162842	.6283	.3223
.952257	2.031580	.6641	.3130
.979113	1.891703	.7216	.2951
.992836	1.809950	.7656	.2784
1.006800	1.723290	.7869	.2693
1.021039	1.631372	.8075	.2595
1.035583	1.534775	.8274	.2492
1.050463	1.434156	.8466	.2382
1.065708	1.330173	.8651	.2267
1.081347	1.223460	.8827	.2144
1.097410	1.114646	.8994	.2015
1.113925	1.004322	.9150	.1879
1.130922	.893190	.9297	.1735
1.148430	.781746	.9431	.1584
1.166481	.670585	.9554	.1425
1.185103	.560284	.9663	.1257
1.204328	.451350	.9759	.1081
1.224189	.344291	.9840	.0894
1.244719	.239581	.9906	.0696
1.265951	.137658	.9956	.0483
1.276842	.087884	.9990	.0238

Table 2—$[P(x_*), \gamma(x_*)]$ table corresponding to $(x, f(x))$ tables for parametric value $a_3 = -20$

x	$x^2 g(x) = f(x)$	$P(x_*)$	$\gamma(x_*)$
.674040	1.297987	.2193	.2660
.730355	1.623616	.3264	.2906
.773337	1.812883	.4123	.2941
.810500	1.898265	.4895	.2903
.836525	1.904802	.5615	.2820
.861838	1.868274	.6132	.2733
.886975	1.793573	.6630	.2628
.912338	1.685449	.7111	.2504
.929535	1.597135	.7572	.2362
.947061	1.497493	.7867	.2255
.964993	1.387952	.8150	.2140
.983402	1.269955	.8420	.2015
1.002357	1.144948	.8676	.1880
1.021922	1.037967	.8915	.1735
1.042163	.879672	.9138	.1559
1.063143	.742232	.9341	.1406
1.084924	.603439	.9518	.1226
1.107572	.464583	.9671	.1033
1.131152	.326954	.9798	.0828
1.155730	.196054	.9895	.0609
1.181375	.062711	.9962	.0373
1.187962	.027789	.9997	.0109

Table 3—[$P(x_*)$, $\gamma(x_*)$] table corresponding to (x, $f(x)$) tables for parametric value $a_2 = -50$

x	$x^2g(x) = f(x)$	$P(x_*)$	$\gamma(x_*)$
.635342	1.033694	.2989	.2150
.709286	1.370562	.3773	.2245
.737620	1.449841	.5048	.2235
.763528	1.482406	.5621	.2185
.788115	1.473913	.6166	.2116
.812091	1.429014	.6687	.2029
.835951	1.352011	.7186	.1927
.860070	1.247106	.7642	.1815
.884750	1.118536	.8091	.1679
.910245	.970601	.8510	.1524
.936785	.807638	.8892	.1350
.964579	.633992	.9230	.1155
.993830	.453927	.9518	.0950
1.024737	.271575	.9746	.0710
1.040874	.180771	.9907	.0443
1.057500	.090840	.9960	.0298
1.066005	.046340	.9992	.0137

References

Chandrasekhar, S. (1949) : "On Heisenberg's Elementary Theory of Turbulence"—*Proc. Roy. Soc. London,* Ser A, 200.

Dutta Gupta, Santa (1975) : Study of Decay Phenomena and Local and Non-Local Effects in Turbulence (Ph.D. dissertation).

Goldstein, S. (1951) : "On the Law od Decay of Homogeneous Isotropic Turbulence and the Theories of the Equilibrium Spectrum and Similarity Spectra"—*Proc. Camb. Phil. Soc.,* 47, 554.

Karman, T. Von (1937) : "On the Statistical Theory of perturbation", *Proc. Nat. Acad. Sci. U.S.A.*, 23, 98.

Lin, C. C. (1953) : A Critical Discussion of Similarity Concepts in Isotropic Turbulence.

Lin, C. C. and Karaman, T. Von (1951) : Advances in Applied Mechanics, II, 14.

Pai Mazumder, Himadri (1975) : Ph.D. dissertation, 207.

Saha, B. R. (1981) : "Spectrum function of Isotropic Turbulence", presented in the Conference of Indian Science Congress Association, Varanasi.

Saha, B. R. (1981) : "Numerical Solutions for the Spectrum Functions relating to Decay of Homogeneous and Isotropic turbulence in case of Large Reynolds Number", presented in the Eighth Casam-Cu National Research Symposium, Calcutta University, March.

DEPARTMENT OF MATHEMATICS
WOMEN'S POLYTECNIC
JODHPUR PARK, CALCUTTA-68.

AND

DEPARTMENT OF SCIENCE
TECHNICAL TEACHERS' TRAINING INSTITUTE
CALCUTTA.

Bull. Cal. Math., 75, 389-392 (1983)

ERROR LOCATING CODES WITH SUB-BLOCKS OF UNEQUAL LENGTHS

BAL KISHAN DASS* and VINOD TYAGI**

(*Received 19 October 1981*)

Abstract : In this paper, we study codes that are capable of detecting random errors within a single sub-block amongst s-sub-blocks of lengths $t_1, t_2, .. , t_s$; $t_1+t_2+ ... +t_s = n$, and locating that sub-block.

Introduction : A new coding technique viz. 'Error Location-Coding' that lies mid-way between error-detection and error-correction was introduced by Wolf and Elspas (1963). In such codes, the block of received digits is regarded as sub-divided into mutually exclusive sub-blocks and errors occuring within a particular sub-block are detected at the receiver and, in addition, the receiver is able to determine which particular sub-block contains errors. Wolf and Elspas (1963) studied such binary codes by considering all the s-sub-blocks of equal lengths.

In this paper, we extend this study to EL-codes over $GF(q)$ by taking the s-sub-blocks of unequal lengths say $t_1, t_2, \dots , t_s$; $t_1+t_2+\dots+t_s = n$.

An *e-EL* code capable of detecting and locating a single sub-block among s-sub-blocks of lengths $t_1, t_2, \dots , t_s$ containing upto e-errors must satisfy the following conditions :

1) The syndrome resulting from the occurence of e or fewer errors within any sub-block must be distinct from the zero syndrome,

2) The syndrome resulting from the occurence of e or fewer errors within a sub-block of length t_i must be different from the syndrome resulting from any combination of e or fewer errors within any *other* sub-block.

In this paper, we derive lower and upper bounds for error locating codes having unequal lengths of the sub-blocks.

A Lower Bound : In the following theorem, we give a lower bound on the number of parity-check digits necessary for the existence of a code that locates an erroneous sub-block containing e or fewer errors.

THEOREM 1. *The number of parity-check digits r required for an e-EL code over* $GF(q)$ *locating a single corrupted sub-block out of the s-sub-blocks of lengths* $t_1, t_2 \dots , t_s$; $t_1+t_2+\dots+t_s = n$, *is bounded from below by*

$$r \geqslant \log_a \left[1+\sum_{j=1}^{s} \sum_{i=1}^{e/2} \binom{t_j}{i} (q-1)^i \right] \qquad \dots \quad (1)$$

Proof: The maximum number of distinct syndromes available using r-check digits is q^r. The proof proceeds by first counting the number of syndromes that are required to be distinct by conditions 1) and 2) and then setting this number less than or equal to q^r.

For detection, the syndrome produced by $e/2$ or fewer errors in a given sub-block must be distinct from any such syndrome likewise resulting from another set of up to $e/2$ errors in the same sub-block or else, there would exist a combination of atmost e-errors in that sub-block resulting in the 'O' syndrome [in volation of condition 1)].

For location, any syndrome produced by a combination of upto e-errors in any sub-block must be distinct from any other combination of upto e-errors in any other sub-block; hence it will certainly be true for syndromes of upto $e/2$ errors whether in the same or in different sub-blocks. Since there are $\binom{t_j}{i}$ combinations of i-errors out of t_j places in the j-th sub-block, therefore, in all these must be

$$1+ \sum_{j=1}^{s} \sum_{i=1}^{e/2} \binom{t_j}{i} (q-1)^i \qquad \dots \quad (2)$$

distinct syndromes including the zero syndrome. The theorem now follows by taking the logarithm.

COROLLARY: *Setting $q=2$ and $t_1 = t_2 = \dots = t_s = t$, the bound in* (1) *reduces to*

$$r \geqslant 1 + s \sum_{i=1}^{e/2} \binom{t}{i} \qquad \dots \quad (3)$$

which is a result due to Wolf and Elspas (1963).

An Upper Bound: The following theorem presents an upper bound on the sufficient number of check-digits required for the existence of a code considered in the preceding section.

THEOREM 2: *An (n, k) linear code over $GF(q)$ capable of detecting e or fewer errors occuring within a single sub-block and locating that sub-block amongst s-sub-blocks of lengths $t_1, t_2, \dots, t_s$; $t_1 + t_2 + \dots + t_s = n$; $e \leqslant t_j$ for all j, can always be constructed using r-check digits where r is the smallest integer satisfying the inequality*

$$r \geqslant \log_q \left[2+ \sum_{i=1}^{e-1} \binom{t_s-1}{i} (q-1)^i + \sum_{p=1}^{s-1} \sum_{j=1}^{e} \binom{t_p}{j} (q-1)^j \left\{ 1+ \sum_{i=1}^{e-1} \binom{t_s-1}{i} (q-1)^i \right\} \right] \qquad \dots \quad (4)$$

Proof : The theorem will be proved by constructing an appropriate parity-check matrix H for the desired code. Let us suppose that we have added $t_1+t_2+\dots+t_{s-1}$ columns corresponding to the first $(s-1)$ sub-blocks and $(j-1)$ columns $h_1, h_2, \dots, h_{j-1}$ to the s-th sub-block of H.

In view of the conditions 1) and 2) stated in the introduction, the j-th column h_j to be added to the s-th sub-block of H must satisfy the following two conditions :

(i) h_j should not be a linear combination of any $e-1$ or fewer columns from amongst $h_1, h_2, \dots, h_{j-1}$.

(ii) h_j should not be a linear combination of any $e-1$ or fewer columns from amongst $h_1, h_2, \dots, h_{j-1}$ together with a linear combination of any e or fewer columns from any one of the $(s-1)$ sub-blocks.

The requirement (1) gives rise to

$$1+\sum_{i=1}^{e-1}\binom{j-1}{i}(q-1)^i \qquad \dots \quad (5)$$

possible linear combinations whereas the requirement (ii) gives rise to

$$\sum_{p=1}^{e-1}\sum_{j=1}^{e}\left(\begin{smallmatrix}t\\j\end{smallmatrix}\,p\right)(q-1)^j\left\{+\sum_{i=1}^{e-1}\binom{j-1}{i}(q-1)^i\right\} \qquad \dots \quad (6)$$

possible linear combinations.

Thus, the total number of linear combinations to which h_j cannot be equal, is

$$1+(5)+(6) \qquad \dots \quad (7)$$

including the zero pattern.

At worst all these linear combinations may yield to a distinct sum. Therefore, a column h_j can be added to H provided that

$$q^r \geqslant (7) \qquad \dots \quad (8)$$

Thus for $j=t_s$, the theorem follows by taking logarithm.

COROLLARY : *For* $q=2$; $t_1=t_2=\dots=t_s=t$, *the bound obtained in* (4) *reduces to*

$$r \geqslant \log_2\left[2+\sum_{i=1}^{e-1}\binom{t}{i}+(s-1)\sum_{j=1}^{e}\binom{t}{j}\left\{1+\sum_{i=1}^{e-1}\binom{t-1}{i}\right\}\right] \qquad \dots \quad (9)$$

This result in the binary case is due to Dass and Muttoo (1981) which is an improvement over the bound given by Wolf and Elspas (1963).

We conclude with an example.

Consider a (11, 6) binary code with parity-check matrix.

$$\left(\begin{array}{cccc|ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

This matrix has been constructed by the synthesis procedure outlined in the proof of theorem 2, by taking $t_1 = 4$, $t_2 = 7$ and $e = 2$. It can be verified that the code so constructed is a 2-*EL* code.

References

Dass, B. K. and S. K. Muttoo, (1981) : "Comments on Error Locating Codes—A New Concept in Error Control", *IEEE Trans. on Inf. Theory*, 27, 1, 136.

Wolf, J. K. and B. Elspas, (1963) : "Error-Locating-Codes-A New Concept in Error Control", *IEEE Trans. on Inf. Theory*, IT-9, 2, 113.

---

*DEPARTMENT OF MATHEMATICS, P.G.D A.V. COLLEGE (UNIVERSITY OF DELHI),
NEHRU NAGAR, NEW DELHI-110 065, INDIA.

**DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DELHI,
DELHI-110 007, INDIA.

Bull. Cal. Math. Soc., **75**, 393-401 (1983)

"DIFFUSION PROCESSES IN A HOMOGENEOUS SHEAR-FREE TURBULENCE UNDER THE ASSUMPTION OF QUASI-NORMAL BEHAVIOUR OF COUPLED VELOCITY—PRESSURE FIELD".

BHULU RANJAN SAHA AND KSHETRA MOHAN GHOSH

(*Received 25 September 1981 ; Revision Received 7 June 1982*)

Abstract : In this paper, we have dealt with the diffusion processes in a homogeneous shear-free turbulence under the assumption of quasi-normal behaviour of coupled velocity-pressure field. For a number of physical phenomena, fluctuations in static pressure are as important as velocity fluctuations and their auto-and-cross-correlations $Q_{p,p'}$ between pressure fluctuations at two neighbouring points have been expressed in terms of velocity correlations from the equation of motion. In the process, we have obtained $f(r, t)$ in the form

$$\overline{aa'} = -\frac{\partial^2}{\partial r^2} Q_{p,p'}(r,t) + \gamma^2 u'^2 \left\{ \frac{\partial^4 f}{\partial r^4} + \frac{8}{r}\frac{\partial^3 f}{\partial r^3} + \frac{6}{r^2}\frac{\partial^2 f}{\partial r^2} - \frac{6}{r^3}\frac{\partial f}{\partial r} \right\}$$

where $a = \frac{dv_2}{dt}$, $a' = \frac{dv'_2}{dt'}$, v_2 and v'_2 being the velocity components at two distinct points.

The Lagrangian auto-correlation co-efficient $R_L(\tau)$ has been calculated to an approximation as

$$R_L(\tau) = e^{-\frac{41}{32}\cdot\frac{\tau^2}{t^2}},$$

for substitution in the fundamental equation of turbulent diffusion

$$\overline{y^2(t)} = 2\,\overline{v_2^2(t)}\,.\int_0^t (t-\tau)\, R_L(\tau)\, d\tau.$$

The asymptotic behaviours, revealed subsequently, give us the following informations :—

(a) for small diffusion time, the diffusion is proportional to time,

(b) for large diffusion time, the diffusion is proportional to the square root of time.

In this paper, the calculation is made with the starting point of correlation between accelerations *i.e.* $\overline{\frac{dv_2}{dt}\cdot\frac{dv'_2}{dt'}}$ instead of correlations of the type $\overline{v_2\, v'_2}$. The final results obtained, so far as the asymptotic behaviours narrated above are concerned, are quite the same.

1. Introduction : In the deduction that follows, let us introduce Lagrangian autocorrelation co-efficient as

$$R_L(\tau) = \frac{\overline{v_2(t)\,.\,v_2(t+\tau)}}{\overline{v_2^2}}\,, \qquad \dots \quad (1)$$

where $v_2(t)$ and $v_2(t+\tau)$ are instantaneous velocities at the point P (say) for two instants t and $t+\tau$. In the context of this correlation co-efficient, there corresponds Lagrangian time-scale of turbulence

$$L_L = \int_0^\infty R_L(\tau)\,d\tau. \qquad \dots \quad (2)$$

As $R_L(\tau)$ is an even function of τ, we have $R_L(\tau) = R_L(-\tau)$ and as an obvious relation, we further obtain $R_L(o) = 1$.

As usual, the fundamental equation of turbulent diffusion is given by

$$\overline{y^2(t)} = \overline{2\,v_2^2(t)} \int_0^t (t-\tau)\,R_L(\tau)\,d\tau \qquad \dots \quad (3)$$

where

$$R_L(\tau) = \frac{\overline{v_2(t')\,.\,v_2(t'-\tau)}}{\overline{v_2^2}} \qquad \dots \quad (4)$$

and

$$t' - t = \tau. \qquad \dots \quad (5)$$

With the help of the equation of motion

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2}{\partial x_l\,\partial x_l} u_i, \qquad \dots \quad (6)$$

we obtain

$$\frac{1}{\rho^2}\,\frac{\partial^4\,\overline{pp'}}{\partial\xi_i\,\partial\xi_i\,\partial\xi_l\,\partial\xi_l} = \frac{\partial^4\,\overline{(u_i u_j)(u'_i u'_j)}}{\partial\xi_i\,\partial\xi_j\,\partial\xi'_i\,\partial\xi'_j} \qquad \dots \quad (7)$$

[Hinze, 1975, p. 306]

Introducing the relation

$$Q_{p,p'} = \frac{1}{\rho^2}\,\overline{pp'} \qquad \dots \quad (8)$$

where $Q_{p\,p'}$ denotes double pressure correlation, we apply Millionshtehikov's quasi-normality hypothesis for $\overline{u_i\, u_j\, u'_i\, u'_j}$ to obtain the relation

$$\frac{1}{r}\,\frac{\partial^4}{\partial r^4}\,(rQ_{p,p'}) = 2\,\frac{\partial^2}{\partial \xi_i\,\partial \xi_k}\,Q_{j,l}\,(r,\,t)\,\cdot\,\frac{\partial^2}{\partial \xi_j\,\partial \xi_l}\,Q_{i,k}\,(r,\,t)\,, \qquad \text{...}\quad (9)$$

[cf. equations (3-291) and (3-294), Hinze, 1975, p. 308]

where $Q_{i,j}\,(r,\,t)$ denotes double velocity correlation.

Expressing the double velocity correlation in terms of the defining scalar $f\,(r,\,t)$, we finally obtain from (9),

$$Q_{p,p'}\,(r,\,t) = 2u'^4 \int_r^\infty \left(r' - \frac{r^2}{r'}\right)\left(\frac{\partial f}{\partial r'}\right)^2 dr' \qquad \text{...}\quad (10)$$

[cf. equation (3-296), Hinze, 1975, p. 308]

Now, the relation (9) gives

$$\frac{\partial^2}{\partial r^2}\,Q_{p,p'}\,(r,\,t) = 4u'^4 \int_r^\infty \frac{1}{r'}\left(\frac{\partial f}{\partial r'}\right)^2 dr'. \qquad \text{...}\quad (11)$$

2. Derivation of the fundamental equation of turbulent diffusion from the equation of motion.

Let us now follow Uberoi and Corrsin (1953), and assume isotropy. So starting from the momentum-balance equation for the velocity component v_2 at a point P, we write

$$\frac{dv_2}{dt} = -\,\frac{1}{\rho}\,\frac{\partial p}{\partial x_2} + \gamma\,\frac{\partial^2 u_2}{\partial x_k\,\partial x_k}\,. \qquad \text{...}\quad (12)$$

Similarly, for the velocity component v'_2 at another point P', we write

$$\frac{dv'_2}{dt'} = -\,\frac{1}{\rho}\,\frac{\partial p'}{\partial x'_2} + \gamma\,\frac{\partial^2 u'_2}{\partial x'_k\,\partial x'_k}\,, \qquad \text{...}\quad (13)$$

where

$$v_2\,(t) = u_2\,(y_k,\,t), \qquad \text{...}\quad (14)$$

$$v'_2 = v_2\,(t+\tau) = u_2\,(y_k + \eta_k,\, t+\tau) \qquad \text{...}\quad (15)$$

and

$$\eta_k = \int_t^{t+\tau} v_k\,(t')\,dt' \qquad \text{...}\quad (16)$$

[cf. equations (5-139a), (5-139b), (5-140), Hinze, 1975, p. 416].

The relation (16) is capable of expansion in a series as

$$\eta_k = \Big[v_k\Big]_{\tau=0} \cdot \tau + \left[\frac{dv_k}{d\tau}\right]_{\tau=0} \cdot \frac{\tau^2}{2} + \cdots \qquad \cdots \quad (17)$$

[cf. equation (5-144), Hinze, 1975, p. 418]

Putting $\eta_k = r$, $v_k = v_2$ in (17), squaring the resultant equation and taking average thereafter, we obtain as the first approximation

$$r^2 = \overline{v_2^2}\, \tau^2 = u'^2\, \tau^2 \qquad \cdots \quad (18)$$

For our mode of calculations, we take help of the aforesaid equations and proceed as follows :

Multiplying (12) by (13) and using the averaging technique, we obtain

$$\overline{\frac{dv_2}{dt} \cdot \frac{dv'_2}{dt'}} = \frac{1}{\rho^2}\, \overline{\frac{\partial p}{\partial x_2} \cdot \frac{\partial p'}{\partial x'_2}} + \gamma^2\, \overline{\frac{\partial^2 u_2}{\partial x_k\, \partial x_k} \cdot \frac{\partial^2 u'_2}{\partial x'_k\, \partial x'_k}} \qquad \cdots \quad (19)$$

where

$$\overline{\frac{\partial p}{\partial x_2} \cdot \frac{\partial^2 u'_2}{\partial x' \;\, \partial x'_k}} = \overline{\frac{\partial p'}{\partial x'_2} \cdot \frac{\partial^2 u_2}{\partial x_k\, \partial x_k}} = O.$$

Let us not introduce the symbols

$$\left.\begin{aligned} \frac{dv_2}{dt} &= a \\ \text{and} \qquad \frac{dv'_2}{dt'} &= a' \end{aligned}\right\} \qquad \cdots \quad (20)$$

So the equation (19) reads as

$$\overline{aa'} = -\frac{1}{\rho^2}\, \frac{\partial^2}{\partial \xi^2}\, \overline{pp'} + \gamma^2\, \frac{\partial^4}{\partial \xi_k\, \partial \xi_k\, \partial \xi_l\, \partial \xi_l}\, \overline{u_2\, u'_2} \qquad \cdots \quad (21)$$

Now,

$$\frac{\partial^4}{\partial \xi_k\, \partial \xi_k\, \partial \xi_l\, \partial \xi_l}\, \overline{u_2\, u'_2}$$

$$= u'^2 \left\{ \frac{\partial^4 f}{\partial r^4} + \frac{8}{r}\, \frac{\partial^3 f}{\partial r^3} + \frac{6}{r^2}\, \frac{\partial^2 f}{\partial r^2} - \frac{6}{r^3}\, \frac{\partial f}{\partial r} \right\}. \qquad \cdots \quad (22)$$

Substituting (8) and (22) in (21), we easily obtain

$$\overline{aa'} = -\frac{\partial^2}{\partial r^2}\, Q_{p,p'}(r, t) + \gamma^2\, u'^2 \left\{ \frac{\partial^4 f}{\partial r^4} + \frac{8}{r}\, \frac{\partial^3 f}{\partial r^3} + \frac{6}{r^2}\, \frac{\partial^2 f}{\partial r^2} - \frac{6}{r^3}\, \frac{\partial f}{\partial r} \right\} \qquad \cdots \quad (23)$$

We can have further

$$\overline{aa'} = \overline{\frac{dv_2}{dt} \cdot \frac{dv'_2}{dt'}} = -\frac{d^2}{d\tau^2}\,\overline{v_2\,v'_2}$$

$$= -\frac{d^2}{d\tau^2}\left\{\overline{v_2^2}\,R_L(\tau)\right\} = -\overline{v_2^2}\frac{d^2}{d\tau^2}R_L(\tau)\,, \quad \ldots \quad (24)$$

where $t'-t=\tau$.

From (22), (23) and (24) one can obtain

$$\overline{v_2^2}\,\frac{d^2}{d\tau^2}R_L(\tau)=\frac{\partial^2}{\partial r^2}\,Q_{p,p'}(r,t)-\nu^2u'^2\left\{\frac{\partial^4 f}{\partial r^4}+\frac{8}{r}\frac{\partial^3 f}{\partial r^3}+\frac{6}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{6}{r^3}\frac{\partial f}{\partial r}\right\} \quad \ldots \quad (25)$$

Integrating (25) with respect to τ, we obtain

$$\overline{v_2^2}\,\frac{d}{d\tau}R_L(\tau)=\left[\frac{\partial^2}{\partial r^2}\,Q_{p,p'}(r,t)-\nu^2u'^2\left\{\frac{\partial^4 f}{\partial r^4}+\frac{8}{r}\frac{\partial^3 f}{\partial r^3}+\frac{6}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{6}{r^3}\frac{\partial f}{\partial r}\right\}\right].\,\tau+A.$$

If $\tau=0$, $A=0$.

So

$$\overline{v_2^2}\,\frac{d}{d\tau}R_L(\tau)=\tau\left[\frac{\partial^2}{\partial r^2}\,Q_{p,p'}(r,t)-\nu^2u'^2\left\{\frac{\partial^4 f}{\partial r^4}+\frac{8}{r}\frac{\partial^3 f}{\partial r^3}+\frac{6}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{6}{r^3}\frac{\partial f}{\partial r}\right\}\right] \quad \ldots \quad (26)$$

Integrating (26) again with respect to τ, we obtain

$$\overline{v_2^2}\,R_L(\tau)=\frac{\tau^2}{2}\left[\frac{\partial^2}{\partial r^2}\,Q_{p,p'}(r,t)-\nu^2u'^2\left\{\frac{\partial^4 f}{\partial r^4}+\frac{8}{r}\frac{\partial^3 f}{\partial r^3}+\frac{6}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{6}{r^3}\frac{\partial f}{\partial r}\right\}\right]+B$$

If $\tau=0$, $R_L(\tau)=1$ and $B=\overline{v_2^2}$.

So,

$$\overline{v_2^2}\,R_L(\tau)=\overline{v_2^2}-\frac{\tau^2}{2}\left[-\frac{\partial^2}{\partial r^2}\,Q_{p,p'}(r,t)+\nu^2u'^2\left\{\frac{\partial^4 f}{\partial r^4}+\frac{8}{r}\frac{\partial^3 f}{\partial r^3}+\frac{6}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{6}{r^3}\frac{\partial f}{\partial r}\right\}\right]$$

i.e.,

$$R_L(\tau)=1-\frac{\tau^2}{2\overline{v_2^2}}\left[-\frac{\partial^2}{\partial r^2}\,Q_{p,p'}(r,t)+\nu^2u'^2\left\{\frac{\partial^4 f}{\partial r^4}+\frac{8}{r}\frac{\partial^3 f}{\partial r^3}+\frac{6}{r^2}\frac{\partial^2 f}{\partial r^2}-\frac{6}{r^3}\frac{\partial f}{\partial r}\right\}\right] \quad \ldots \quad (27)$$

Substituting (11) in (27), we obtain

$$R_L(\tau)=1-\frac{\tau^2}{2\,\overline{v_2^2}}\left[4u'^4\int_r^\infty \frac{1}{r}\left(\frac{\partial f}{\partial r}\right)^2 dr+\nu^2u'^2\left\{\frac{\partial^4 f}{\partial r^4}+\frac{8}{r}\,\frac{\partial^3 f}{\partial r^3}+\frac{6}{r^2}\,\frac{\partial^2 f}{\partial r^2}-\frac{6}{r^3}\,\frac{\partial f}{\partial r}\right\}\right] \quad \dots \quad (28)$$

Without any loss of generality, we can write

$$u'^2=\overline{v_2^2}=\frac{\nu}{t}\,. \quad \dots \quad (29)$$

Also we write

$$f=e^{-r^2/8\nu t} \quad \dots \quad (30)$$

[cf. equation (3-45), Hinze, 1975, p. 201].

When (30) is used in (28), we obtain

$$R_L(\tau)=1-\frac{\tau^2}{2\overline{v_2^2}}\left[\frac{u'^4}{2\nu t}\;e^{-r^2/4\nu t}+u'^2\left(\frac{33}{16t^2}-\frac{7r^2}{32\nu t^3}+\frac{r^4}{256\nu^2t^4}\right)e^{-r^2/8\nu t}\right]$$

$$=1-\frac{\tau^2}{2\,\overline{v_2^2}}\left[\frac{u'^4}{2\nu t}\left(1-\frac{r^2}{4\nu t}+\frac{1}{\underline{|2}}\cdot\frac{r^4}{16\,\nu^2t^2}-\frac{1}{\underline{|3}}\cdot\frac{r^6}{64\,\nu^3t^3}+\dots\right)\right.$$

$$\left.+\,u'^2\left(\frac{33}{16t^2}-\frac{7r^2}{32\nu t^3}+\frac{r^4}{256\nu^2t^4}\right)\left(1-\frac{r^2}{8\nu t}+\frac{1}{\underline{|2}}\cdot\frac{r^4}{64\nu^2t^2}-\frac{1}{\underline{|3}}\cdot\frac{r^6}{512\nu^3t^3}+\dots\right)\right] \quad \dots \quad (31)$$

Now from (31) it is apparent that if we substitute $r^2=\overline{v_2^2}\,.\,\tau^2=u'^2\tau^2$ from (18) and make use of further relation $u'^2=\overline{v_2^2}=\dfrac{\nu}{t}$ from (29), we obtain a relationship of the form

$$R_L(\tau)=1+a\,.\,\tau^2+b\,.\,\tau^4+\dots\dots\dots$$

for suitable calculated values of a,b, etc. For $\tau \to 0$, when we retain the terms only upto τ^2, we get the form function of $R_L(\tau)$ giving us a parabolic behaviour of $R_L(\tau)$ when $\tau \to 0$.

Thus, we obtain a relationship which may be considered as the approximation of an exponential function to the exclusion of the parabolic behaviour of $R_L(\tau)$ when $\tau \to 0$. So for large Reynolds number, we write the resultant form of (31) as

$$R_L(\tau)=e^{-\frac{41}{32}\,.\,\frac{\tau^2}{t^2}}. \quad \dots \quad (32)$$

Now, equation (3) can be re-written as

$$\overline{y^2(t)} = 2\,\overline{v_2^2(t)} \int_0^t (t-\tau)\, e^{-\frac{41}{32}\cdot\frac{\tau^2}{t^2}}\, d\tau. \qquad \ldots \quad (33)$$

The relation (33) is known as the fundamental equation of turbulent diffusion derived from the equation of motion *i.e.* from the momentum-balance equation for the velocity component v_2 relating to the point P for relevant time $t+\tau$.

Substituting (32) in (2), we thus obtain

$$L_L = \int_0^\infty e^{-\frac{41}{32}\cdot\frac{\tau^2}{t^2}}\, d\tau$$

$$= \sqrt{\frac{8\pi}{41}}\,.\,t \qquad \ldots \quad (34)$$

Equation (34) shows that L_L is a function of t. It is to be noted in this context that the above relation for giving L_L in terms of t is not true when t is very large compared to L_L. On the contrary, for the calculation of $\overline{y^2}$, we are to make the asympototic calculation for dispersion time t very large compared to Lagrangian scale of turbulence, (Frenkiel, 1953).

3. Asymptotic behaviour of the equation (33) for short diffusion time as well as for large diffusion time.

a) Asymptotic behaviour for short diffusion time :

Equation (32) can be expanded in a power series, and terms in $\frac{\tau}{t}$ of an order higher than the second are neglected.

Hence $R_L(\tau)$ is written as

$$R_L(\tau) = 1 - \frac{41}{32}\cdot\frac{\tau^2}{t^2}. \qquad \ldots \quad (35)$$

Substituting (35) in (33), we obtain

$$\overline{y^2(t)} = 2\,\overline{v_2^2(t)} \int_0^t (t-\tau)\left(1 - \frac{41}{32}\cdot\frac{\tau^2}{t^2}\right) d\tau$$

$$= 2\,\overline{v_2^2(t)} \int_0^t (t-\tau)\, d\tau - \frac{41}{16t^2}\,\overline{v_2^2} \int_0^t (t-\tau)\,\tau^2\, dt$$

113619

$$= \overline{v_2^2} \cdot t^2 - \frac{41}{16t^2}\, \overline{v_2^2} \left\{ t \cdot \frac{t^3}{3} - \frac{t^4}{4} \right\}$$

$$= \overline{v_2^2} \cdot t^2 - \frac{41}{16}\, \overline{v_2^2} \cdot t^2 \cdot \frac{1}{12} = \overline{v_2^2} \cdot t^2 \left(1 - \frac{41}{192}\right)$$

or,

$$\overline{y^2} \approx \overline{v_2^2} \cdot t^2, \; (t << L_L) \qquad \dots \quad (36)$$

i.e. in alternative notation,

$$y'^2 \approx v_2'^2 \cdot t^2$$

or,

$$y' \approx v_2' \cdot t, \qquad (t << L_L). \qquad \dots \quad (37)$$

This shows that for small diffusion time, the diffusion is proportional to time.

b) Asymptotic behaviour for large diffusion time :

When the dispersion time is large compared to the Lagrangian scale of turbulence L_L, we can write the equation (33) as

$$\overline{y^2(t)} = 2\,\overline{v_2^2} \cdot t \int_0^\infty e^{-\frac{41}{32} \cdot \frac{\tau^2}{t^2}}\, d\tau - 2\,\overline{v_2^2} \int_0^\infty \tau \cdot e^{-\frac{41}{32} \cdot \frac{\tau^2}{t^2}}\, d\tau$$

$$= 2\,\overline{v_2^2} \cdot t \cdot L_L - 2\,\overline{v_2^2} \cdot \frac{16t^2}{41}$$

$$= 2\,\overline{v_2^2} \cdot L_L \cdot t - \frac{32}{41} \cdot \overline{v_2^2} \cdot t^2 . \qquad \dots \quad (38)$$

For the variance $\overline{y^2}$, the second term on the right hand side of the equation (38) is a constant and may then be neglected. Then equation (38) reads as

$$\overline{y^2} = 2\,\overline{v_2^2} \cdot L_L \cdot t, \qquad (t \gg L_L) \qquad \dots \quad (39)$$

i.e. in alternative notation,

$$y'^2 \approx v_2'^2 \cdot L_L \cdot t$$

or,

$$y' \approx v_2' \cdot \sqrt{L_L} \cdot \sqrt{t}, \qquad (t \gg L_L) \qquad \dots \quad (40)$$

[cf. remarks at the end of (34)]

The relation (40) shows that for large dispersion time, the diffusion is proportional to the square root of time.

Remarks 1 : It is to be noted that in our calculation, we have started with correlation of accelerations instead of correlation of velocities and still arrive at the same asymptotic relations which were known before-hand, *viz.* a) for small diffusion time, the diffusion is proportional to time, b) for large diffusion time, the diffusion is proportional to the square root of time.

2) The equation (36) corresponds to the variance $\overline{y^2}$ when the time is very short compared with the Lagrangian time-scale of turbulence L_L. Equation (39) corresponds to the variance $\overline{y^2}$ when the time is very large compared with the Lagrangian time-scale of turbulence L_L. All these show that for short diffusion times, all frequencies contribute to $\overline{y^2}$, whereas for large diffusion times, only the low frequencies contribute to $\overline{y^2}$.

References

Frenkiel, F. N. (1953) : "*Advances in Applied Mechanics*", 3. Academic Press, New York.

Hinze, J. O. (1975) : *Turbulence*, Mc. Graw-Hill Book Co.

Uberoi, M. S. and Corrsin, S. (1953) : "Diffusion of heat from a line source in isotropic turbulence". *Natl. Advisory Comm. Aeronaut. Tech.* Repts. No. **1142**.

पुस्तकालय
113619
गुरुकुल कांगड़ी

DEPARTMENT OF MATHEMATICS
WOMEN'S POLYTECHNIC
JODHPUR PARK,
CALCUTTA-700 068
AND
DEPARTMENT OF SCIENCE
TECHNICAL TEACHERS' TRAINING INSTITUTE
CALCUTTA.

Compiled
1999-2008

5/1/07